Do Magic Rectangles Exist?

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Abstract

A possibility of magic rectangles existence is considered. It is proven that magic rectangles with positive integers do not exist. The magic rectangles with both negative and positive numbers are shown exist and their properties are studied. The ways of building magic rectangles were proposed for both Type A and Type B.

Introduction

Over the last year I always saw magic squares [1,2] in Math Club at school and Math competitions, such as Quest Academy Math Madness, but I've never seen magic rectangles. I was curious whether magic rectangles even exist and it led me to this project.

Body

Definitions

Before starting with this project, let's define two types of magic rectangles if they exist: Type A and Type B. *A magic rectangle of Type A* is a rectangle where all distinct numbers in every column and row add up to the same, so-called, magic sum (*S*). *A magic rectangle of Type B* is a rectangle where all distinct numbers in every column, row, and two diagonals add up to the same magic sum (S). The reason I started with Type A is that the search for Type B magic rectangles seemed to be very challenging for me.

Magic sum is 0

Let's count all numbers in a magic $n \times m$ rectangle by rows and then do the same but by columns. We should get the same number or if we use S as the magic sum:

$$n \times S = m \times S$$

This equation is true only if $S=0$, as n and m are different numbers. So, if magic rectangles exist, the magic sum (S) has to be 0.

Magic rectangle with positive integers doesn't exist

Any magic rectangle would have at least one row or column with 2 or more numbers. Because the magic sum is 0 and if we put our first positive integer in one of the cells, we need to use negative integer to make the sum equal 0 but it's impossible. Let's consider not only positive integers but all integers including negative integers.

1 × n rectangles do not exist

If we count the rows it could have the sum of 0. But when you count the columns, the only way to make all sums in columns 0 is to use more than one 0, but we can't do it as all numbers must be distinct.

2x3 magic rectangle exist! Eureka!

When I tried different numbers for 2×3 magic rectangle, I've finally found one! Eureka!

2	3	-5
-2	-3	5

There are infinite number of magic rectangles

Let's try to come up with generalization and I found one:

-k	-1	(k+1)
k	1	-(k+1)

Because *k* has infinity of different values, there is infinity of 2×3 magic rectangles (Type A).

Bigger magic rectangles (Type A)

After many tries and errors I found a 3×4 magic rectangle.

-2	2	4	-4
7	-7	-1	1
-5	5	-3	3

A general way to build any n × m magic rectangle for even n.

Also I found a way to build any magic $n \times m$ rectangle for even n. First I start with putting 1 in the top left corner.

1					

Then I move down and put -1 under 1. Then I put 2 and -2, and so on until the last column.

1	3	5	7	9	
-1	-3	-5	-7	-9	
2	4	6	8	10	
-2	-4	-6	-8	-10	

In the last column I put the number that make 0 for each row.

1	3	5	7	9	-25
-1	-3	-5	-7	-9	25
2	4	6	8	10	-30
-2	-4	-6	-8	-10	30

This method works only if at least either n or m is even. If both n and m are odd, I couldn't find any way to build a magic rectangle or prove that it's not possible.

Illusive magic rectangles of Type B

Now, when I found a few magic rectangles of Type A and studied some of their properties, I was really curious whether I can find any magic rectangle of Type B or whether it's even possible. So far, all my magic rectangles of Type A were not magic rectangles of Type B. In other words, the diagonals do not add up to 0.

2×3 magic rectangles of Type B don't exist

First of all, I tried to find a 2×3 magic rectangle of Type B. It turned out that they do not exist and here is why.

X	X	b
a	X	X

All cells with *X* represent one of the diagonals and add up to 0. It means that $a+b$ is also 0. It's possible only if $b = -a$, but the *X* above *a* must be $-a$ too, so the first column adds up to 0 but we can't have two $-a$. So, it's impossible.

A 3×4 magic rectangle of Type B exist! Eureka!

After many tries, I finally found a magic rectangle where each row, each column, and each diagonal adds up to 0.

8	-5	-2	-1
-9	3	-3	9
1	2	5	-8

How to build any nxm magic rectangle

I noticed a pattern in building a 3×4 magic rectangle of Type B that you need to put pairs of positive and negative integers that makes 0 into symmetrical cells relatively to the center of the rectangle. I believe in doing so for bigger rectangles, I can build any $n \times m$ magic rectangle of Type B, although I can't prove it. At the end, I build a 3×5 magic rectangle with 0 in the center.

-38	8	-1	10	21
59	2	0	-2	-59
-21	-10	1	8	38

Conclusion

I proved that magic rectangles do not exist for positive numbers and if they exist, the magic sum has to be 0. For positive and negative numbers I found magic rectangles and explained a few ways of building any $n \times m$ magic rectangle of Type A and Type B. I still can't offer a reliable way of building any $n \times m$ magic rectangle of Type B and it would be a great direction for the future study. It would be also interesting to figure out whether it's possible to find a magic rectangle without any positive-negative pair.

Bibliography

1. https://en.wikipedia.org/wiki/Magic_square
2. http://mathforum.org/alejandre/magic.square/adler/adler.whatsquare.html