QED Paper

Frogs on a Log

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Table of Contents

Purpose

We are researching the mathematical game most commonly "Frogs on a Log." We wanted to find a formula for the minimum number of moves based on any amount of frogs on each side, and we wanted to come up with several variations.

Procedure

We first compiled any data we could. We made a table with the number of frogs on one side and the moves on the other. With this data, we came up with a formula.

Upon coming up with the formula, we then made three variations and used a similar process of thinking and came up with formulas for the variations.

Conclusion

To conclude, the biggest thing we learned was how to solve a problem: by compiling data and finding the formula for the original problem, then extending it further and learning from what we did wrong.

Abstract

We are researching the game "Frogs on a Log." The setup of the game is a pegboard with a 1x9 row of holes. In the holes are 4 pegs on the right and 4 pegs on the left. There is an empty hole in the middle. The objective of the game is to get the pieces on the left to the places on the right and the frogs on the right to the left. However, a frog can only move in one direction. A frog on the left can only move to the right, and the frogs on the right can only move to the left. A frog can also jump over one frog directly in front of it. We planned to explore a number of ways to win this game and how to find a formulated solution to solving it. We also want to learn how to analyze any math puzzle and know how to solve it algebraically. Any puzzle has its solution, and we want to learn how to find the solution to ours.

Introduction

The game (by its original name "Toads and Frogs") was first published by Richard K. Guy. The game was introduced in Guy's book *Winning Ways for your Mathematical Plays* in 1982. However, a variation to the problem was first seen in an engraving of the Princess of Soubise done by Claude Auguste Berey in 1687. According to legend, the French mathematician Pelisson invented the problem to entertain King Louis XIV, but this is not confirmed. The problem is not very famous, but it is harder than it looks.

Body

We started the puzzle by simply playing around. We soon realized that there was only one specific way to solve the puzzle, so any solution would be the solution with the minimum number of moves. Using this information, we just started solving as best as we could.

We found that the only way to solve it was to make a scenario where the colored pegs alternated between colors (for example, left-right-left-right.) We knew this because if there were two pegs of the same side in a row, then the other type of peg could not jump between them or jump over them both, so they couldn't get to the other side. We tried solving the puzzle with different amounts of pegs on each side and recorded our answers in the table below.

Frogs on each side (*F*)	Moves (*M*)
1	3
2	8
3	15
4	24
8	80

Then, we made the conjecture listed below:

Frogs on each side	Change	Moves
1	*3 =	3
2	*4 =	8
3	*5 =	15
4	*6 =	24
8	*10 =	80

And then we made yet another conjecture:

Frogs on each side (*F*)	Change	Moves (*M*)
1	*(1+2) =	3
2	*(2+2) =	8
3	*(3+2) =	15
4	*(4+2) =	24
8	*(8+2) =	80

We then came up with the formula $F(F+2) = M$. Distributing F, this gave us F^2+2F. We tested our formula with a randomly large amount. We tried 14 and plugged in the formula. We got 224. We played the game with 14 pegs, and the number of moves it took matched our number of 224.

To show our work, we created a notation for solving the problem in an organized way. We used L for the pieces on the left side of the board, R for the pieces on the right. R_1 represents the frog on the very right. R_2 represents the spot next to R_1. L_1 represents the frog to the very left, and so on, as shown below in fig. 1. P_1 represents the spot on the very left. P_2 represents the spot to the right of the very left, as shown below in fig. 2.→ represents a move directly to the left. ← represents a move to the right. ↶ represents a jump to the left. ↷ represents a jump to the right.

L_1	L_2	L_3		R_3	R_2	R_1

Fig. 1

P_1	P_2	P_3	P_4	P_5	P_6	P_7

Fig. 2

1. $R_3 \leftarrow P_4$
2. $L_3 \curvearrowright P_5$
3. $L_2 \rightarrow P_3$
4. $R_3 \curvearrowleft P_2$
5. $R_2 \curvearrowleft P_4$
6. $R_1 \leftarrow P_6$
7. $L_3 \curvearrowright P_7$
8. $L_2 \curvearrowright P_5$
9. $L_1 \curvearrowright P_3$
10. $R_3 \leftarrow P_1$
11. $R_2 \curvearrowleft P_2$
12. $R_1 \curvearrowleft P_4$
13. $L_2 \rightarrow P_6$
14. $L_1 \curvearrowright P_5$
15. $R_1 \leftarrow P_3$

And with the above notation, the puzzle is finished.

Logically, this formula works because the total amount of moves to finish the puzzle is the number of slides + the number of jumps. Each frog has to jump over each frog from the other side, so this can be simplified to F^2. The first frog also has to move $F + 1$ spaces to end up in space $F + 2$, the second frog has to move $F + 1$ spaces to end up in space $F + 3$, and so on and so forth. Added together, all the frogs have to move $F(F + 1)$ spaces to their final positions. By similar reasoning, the frogs on the other side have to move $F(F + 1)$ spaces to their final positions as well. Added together, all the frogs have to move $2(F(F + 1))$ spaces. Since one slide covers two cells, and there are F^2 of them, the number of slides is equal to $2(F(F + 1)) - 2F^2 = 2F$. Plugging this back into our original formula of slides + jumps, we get F^2+2F.

We also ran several of our own variations on the puzzle. One of the variations was having more frogs on one side than the other.

Below, we made a table similar to the first.

Frogs on the left side (L)	Frogs on the right side (R)	Moves (M)
1	6	13
1	2	5
----------------------------------	----------------------------------	----------------------------------
2	5	17
2	3	11
----------------------------------	----------------------------------	----------------------------------
3	4	19
----------------------------------	----------------------------------	----------------------------------
4	5	29

We came up with the formula (L+R)+(L*R) by studying the table. It became clear that the formula not only worked, but it was logical as well. This formula is essentially the same as the first formula when the sides were equal, but we incorporated the different sides.

Another variation we came up with was a scenario where there were more spaces in the middle. In the original problem, there was one empty space separating the two sides of frogs, but we added more spaces and tried to come up with another formula.

Frogs on each side (*F*)	Spaces in the middle (*S*)	Moves (*M*)
2	2	12
2	3	16
2	4	20
3	2	21
3	3	27
3	4	33
4	2	32
4	3	40
4	4	48

We noticed that by doubling S and adding it to F, then multiplying the result by F, it would result in the minimum moves. For example, ((3*2) + 4)*4 = 40, and that lines up with the minimum number of moves. Putting this algebraically, the final formula for this variation of the problem turns out to be $M = F(F + 2S)$.

This works logically because each frog has to jump over each frog on the other side, but each frog also has to 3slide some amount of extra spaces in the middle, so this means that each frog will move more based on how many squares are in the middle.

The final variation to our problem is a combination of both previous variants. We varied the number of frogs on both sides and the spaces in between and tried to come up with a formula that branches over these variations of the puzzle.

Frogs on Left (L)	Frogs on Right (R)	Spaces in between (S)	Moves (M)
3	2	2	16
3	2	3	21
3	2	4	26
4	3	2	26
4	3	3	35
4	3	4	40
4	2	2	20
4	2	3	26
4	2	4	31

We looked at each number and also looked at the previous formulas. We guessed around and found that by multiplying each of the numbers to each other and adding the products of all 3 equations, the result would make our final number of moves. We believe this works logically because each L piece has to jump over each R piece, hence the L*R and each R piece has to jump over each middle square, hence the R*S and each L piece has to jump over each S, hence the L*S. Adding each value, we get (L*R) + (R*S) + (L*S).

Conclusion

To conclude, we ran three variations on a little-known problem most commonly called "Frogs on a Log." We found a formula for every variation we came up with. We varied the amount of space in between frogs, the number of frogs on each side, and a combination of both previous variations. We learned mostly about making formulas for diverse situations that we have never seen, and also about puzzle-solving methods. We developed a strategy for solving any puzzle, which was to play the game while keeping track of movements and results, and then record and come up with a formula for solving the problem. We applied the mentioned strategy to this problem, and it worked very well. In the future, we want to make more variations on the problem. An example of a further variation could be having more than one space in between the pegs on each side. There are a multitude of variations we did not look into that we could definitely try in the future.

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