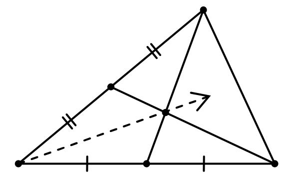
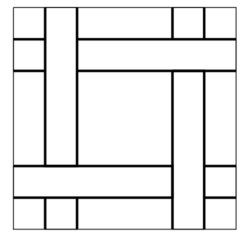
Some Research Problems

Middle School Level (sort of)

A *median* of a triangle is a segment from a vertex of the triangle to the midpoint of the opposite side. Draw two medians of a triangle, and then draw a ray from the third vertex through the medians' point of intersection. Where does it intersect the third side? What if you changed the medians into segments that intersect the opposite side in places 1/3 of the way from one vertex to the other? Other fractions? Does it matter what kind of triangle you start with?



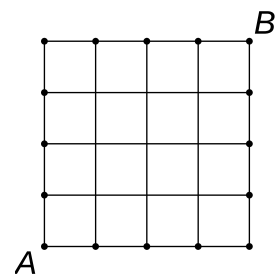
A 7×7 square tile is overlaid with four 1×7 strips in the pattern shown at right. What fraction of the tile's area is covered? Does it matter how the strips are placed? Generalize: if an $n \times n$ tile is covered with four $1 \times n$ strips, what fraction of the tile's area is covered? What about four $m \times n$ strips? What if you use k^2 strips, where $k \geq 2$?



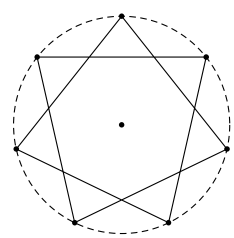
Is it possible to make an equilateral triangle on a geoboard? More formally: is it possible that three points whose coordinates are all integers can form an equilateral triangle? What about in three dimensions? What about other regular polygons? Turn the problem around: suppose you have a lattice of equilateral triangles—is it possible to make a square?

A traditional problem asks how many paths there are from A to B that travel only up and right. Some generalizations and extensions:

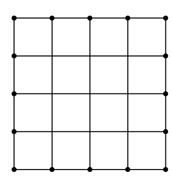
- Is there a general formula for $n \times n$ grids? $m \times n$?
- What if the grid is made of equilateral triangles? Regular hexagons? Other tessellating polygons? (You have to determine reasonable rules for allowed directions in these cases.)
- Suppose two people leave A at the same time, arrive at B at the same time, and walk at the same rate. What is the probability that they meet at a corner somewhere in between?



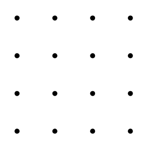
If you arrange 7 points around a circle and then connect them with segments, skipping every other point, you get a star—specifically, the {7,2} star polygon. Notice that we don't count places where the sides intersect as "vertices". What is the sum of the polygon's angles? Does it change if the spaces between the vertices are unequal? (Does it matter if all the vertices lie on a circle?) In what cases does an $\{n, 2\}$ polygon exist? Is there a formula for its angle sum? What about for values besides 2?



Many people are surprised when they are told that the 4×4 grid pictured at right actually contains 30 squares, not just 16. Is there a way to calculate the number of squares for grids of other sizes? $n \times n$? $n \times m$? What if we take the gridlines away, and only require that the vertices lie on lattice points, as shown at far right?



4 × 4 grid



5 × 5 lattice

How many toothpicks does it take to make a 4×4 grid? $n \times n$? $m \times n$? What about in three dimensions—how many toothpicks, or how many square tiles?

A camel needs to cross a desert 300 miles wide. The camel eats one banana every mile he travels, and the camel can only carry 100 bananas at a time. The camel can deposit caches of bananas to be eaten later. For example, the camel could leave carrying 20 bananas, go out 5 miles, deposit 10 bananas, and then go back 5 miles; the next time the camel is at the five-mile mark, he can pick up the 10 bananas that he left there. Is it possible to cross the desert in this fashion? What is the minimum number of bananas needed? What is the minimum number of miles total that the camel needs to walk? Is it possible to cross the desert if the camel's capacity is less than 100 bananas? For what values of n and c is it possible for the camel to cross a desert of width n carrying at most c bananas per trip? Would it make a difference if we used pounds of rice instead of bananas?

According to the 2010 census, Illinois' population was 12,830,632, while the U.S. population was 308,745,538. The U.S. Congress has 435 representatives, and according to the Constitution, representatives are supposed to be allotted according to each state's population. How many representatives should Illinois get? Would your method work for all 50 states?

High School Level (sort of)

For what values of a, b, c does the cubic $x^3 + ax^2 + bx + c$ have exactly two distinct real roots? For what values of a, b, c, d does the quartic $x^4 + ax^3 + bx^2 + cx + d$ have exactly two distinct real roots?

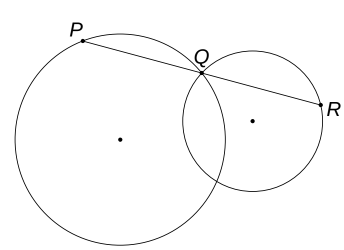
A certain one-celled organism has a reproductive span of one hour: that is, every hour, it either splits into two bacteria, or it dies. The chance of splitting into two bacteria is exactly 50%. How many bacteria should you expect to see after 10 hours? Then generalize: change the probability of splitting, the number of possible "children", or the number of hours. Is it possible to come up with simple functions describing these possibilities?

Suppose that an integer n can be expressed as the sum of two squares; for example, $13 = 2^2 + 3^2$. Notice that $2 \cdot 13 = 26$ can also be expressed as the sum of two squares: $26 = 1^2 + 5^2$. Is that always true? What other multiples of n can be expressed as the sum of two squares?

How many non-congruent triangles with integer sidelengths have a perimeter of 6? Of 7? Of n? What about quadrilaterals?

A quadrilateral is *cyclic* if it's possible to draw a circle passing through all four of its vertices. For integers a, b, c, d, is there always a cyclic quadrilateral with sidelengths of a, b, c, d in that order? Is there more than one? Suppose such a quadrilateral exists for some quadruple (a,b,c,d). If the sidelengths are permuted, must a cyclic quadrilateral exist with sidelengths in the new order? If so, in what other ways are the two quadrilaterals alike?

In the diagram at right, circles with centers 12 units apart have radius 8 and 6, respectively. At Q, one of the points of intersection, a line is drawn in such a way that chords $\overline{PQ}$ and $\overline{QR}$ are of equal length. Compute the length PQ. Then generalize: suppose the centers are d units apart with radii a and b. What is the length PQ? Is it always possible to draw such a line? [Original problem 1983 AIME]



A function f is defined for all real numbers and satisfies the two conditions (i) $f(2 + x) = f(2 - x)$ and (ii) $f(7 + x) = f(7 - x)$, for all real numbers x. If $f(0) = 0$, what other integers n must be solutions to the equation $f(n) = 0$? What if 2 and 7 are replaced by a and b, respectively? [Original problem AIME 1984]

Nine people sit down for dinner where there are three choices of meals. Three people order the beef meal, three order the chicken meal, and three order the fish meal. The waiter serves the nine meals in random order. Find the number of ways in which the waiter could serve the meal types to the nine people so that exactly one person receives the type of meal ordered by that person. Generalize: what if there are $3k$ people and k people order each type of meal? What if there are m types of meals and $3m$ people? km people? [Original problem AIME I 2012]

Butch and Sundance need to get out of Dodge. To travel as quickly as possible, each alternates walking and riding their only horse, Sparky, as follows. Butch begins by walking while Sundance rides. When Sundance reaches the first of the hitching posts that are conveniently located at one-mile intervals along their route, he ties Sparky to the post and begins walking. When Butch reaches Sparky, he rides until he passes Sundance, then leaves Sparky at the next hitching post and resumes walking, and they continue in this manner. Sparky, Butch, and Sundance walk at 6, 4, and 2.5 miles per hour, respectively. The first time Butch and Sundance meet at a milepost, they are n miles from Dodge, and they have been traveling for t minutes. Find $n + t$. Then generalize: what if they walk at a, b and c miles per hour, where $a > b > c$?

It's well known that the continued fraction

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}$$

approaches $\frac{1+\sqrt{5}}{2}$. What happens if all the non-numerator 1's are replaced by 2's:

$$2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}=?$$

Is there a simple formula for the value of such fractions? What if the digits alternate between 2 or more values? Is there a straightforward way to compute continued fractions for any given irrational number? Suppose such a fraction is cut off after a finite number of steps, yielding a "convergent" fraction. What properties do the convergents have? How are they related to the ultimate value of the infinite fraction? [Most of these questions have known answers, but they're really fascinating. A fantastic source is *Continued Fractions,* by Carl Olds, which is out of print but available through online sellers.]