Some Problems Leading to Interesting Mathematical Research

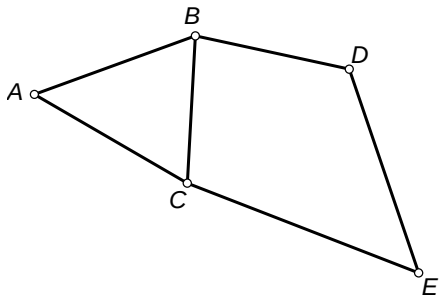
Pure Mathematics

1. The game of *Skunk* (also known as *Pig*) is played with two dice. On each turn, the player rolls the dice over and over again, adding the numbers, until he or she decides to stop or until he or she rolls a 1 on either die. If he or she stops, the total is added to his score, but if he rolls a 1, he gets a zero for the turn.

Play the game with a partner. What are the possible strategies? Which strategies seem to work the best?

2. *Crossing Rivers* is also played with two dice. At the start of the game, each player arranges 12 counters (representing boats) in spaces (representing piers) numbered 2-11 (obviously, it's okay to have more than one counter in a space). At each turn, the dice are rolled, and each player is allowed to remove one boat from the corresponding pier, if there are any—that boat gets to cross the river. If there are no boats on the corresponding pier, the player does nothing. The winner is the first player to remove all of his or her boats.

3. *Cops and Robbers* is played with two pieces of paper, one of which has a diagram showing a network of hideouts and tunnels (as shown below). One player is the robber; the other is the cop, who starts with a fixed number of deputies. The robber starts by selecting a hideout secretly and writing it down on the second sheet of paper. Then the cop player has his deputies search hideouts, one hideout per deputy. If the robber's hideout is one of those searched, then the robber is caught and the cop wins. Otherwise, the robber must move along a (single) tunnel to an adjacent hideout , and the cops search again. The robber is allowed to select one of the previously-searched hideouts, so long as it's connected by a tunnel; unlike the robber, the cops are *not* constrained by tunnels, and can choose any hideouts to search regardless of whether they are connected to previously-searched hideouts. After ten days (turns) of hiding and searching, the cops get discouraged and the robber wins.



Example with 2 deputies.

On the first turn, the robber starts by hiding out in A; the cops search B & C, and don't find the robber. On the second turn, the robber must move, but can only reach B and C; the robber chooses C. On this turn, the cops search A and D. Now the robber has three choices (A, B, or E) and chooses B again. The cops search C and E. The game continues as shown below.

Turn	1	2	3	4	5	6	7
Robber Hideout	A	C	B	D	E	C	A
Searched Hideouts	B & C	A & D	C & E	A & B	C & D	B & E	C & A

Questions to ask:

- → On the map shown, is it always possible for two cops to catch the robber? (Is there a strategy that guarantees capture?)
- → Can one cop guarantee capture on this map? On any map?
- → On what maps is one cop sufficient to guarantee capture within 10 turns? Within any number of turns?
- → Which strategies minimize the number of *workdays* spent by cops (rather than absolute time)?

3. *Zombies vs. Human* is similar to Cops and Robbers, except that (a) both players move along tunnels and (b) players' moves are simultaneous. That is, the human survivor and the zombies determine where they will move, going along a tunnel each time; if they wind up at the same vertex, or use the same tunnel during the same turn, the zombies successfully capture and eat the humans. After ten turns, the zombies get bored and leave the area.

4. *Mad Vet Problems.* A Mad Veterinarian has created three animal transmogrifying machines. Place a cat in the input bin of the first machine, press the button, and .... whirrr .... bing! Open the output bins to find four dogs and one mouse. The second machine can convert a dog into three cats and three mice, and the third machine can convert a mouse into a cat and a dog. Each machine can also operate in reverse (e.g., if you've got four
dogs and one mouse, you can convert them into a cat.)

This Mad Vet has one cat.

- Can he convert it into seven mice? (If so, show how. If not, explain why not. "Seven mice" means exactly seven mice, with no dogs or cats left over.)
- Can he convert it into a kennel of dogs, any size, with no cats or mice left over?
- What starting sets of animals can be converted to all cats, if any?

Applied Mathematics/Modeling

1. *Renting a Car with Gas*

When you rent a car, the car rental company offers you the option of pre-paying for the tank of gas, returning the car with the gas tank full, or paying the rental company to fill up the returned car for you. Ordinarily, the price charged per gallon when you pre-pay is slightly lower than the prices at nearby gas stations, while the price for filling up the returned car is substantially *higher*. For example, if the current gas price is $4.00/gallon, the pre-purchase price might be $3.90 per gallon, while refueling at the end might cost $5.50 per gallon. In general, the rental won't charge you for refueling if the car is less than one gallon short of "full".

Develop a model that allows you to determine the best strategy (a) when you rent the car and (b) when you are getting ready to return it. Your model should take into account that it's not always possible to time your return to fill up the tank yourself, and that "nearby" gas stations may not be as close as you want.

2. *How Much Gas Should I Buy This Week?*(HiMCM 2012)

Gas prices fluctuate significantly from week to week. Consumers would like to know whether to fill up the tank (gas price is likely to go up in the coming week) or buy a half tank (gas price is likely to go down in the coming week).

Consider the following cases:

- Consumer drives 100 miles per week
- Consumer drives 200 miles per week

Assume:

- Gas tank holds 16 gallons and average mileage is 25 miles/gallon => 400 miles/tank
- Consumer buys gas once a week

Therefore, the consumer can drive for 2 weeks or 1 week on half a tank of gas for cases (1) and (2) respectively. Thus the choice each week is whether to buy a full or half tank of gas or no gas.
Use the weekly gas price data available by region, state and city at:

http://www.eia.gov/dnav/pet/pet_pri_gnd_a_epmr_pte_dpgal_w.htm

You can also use weekly crude price data (http://www.eia.gov/dnav/pet/pet_pri_spt_s1_w.htm) and any other publicly available data such as weather data, economic data, world events, etc.

3. *Smoky Mountain Elk* (HiMCM 2012)

You are in charge of a program to reintroduce elk to Great Smoky Mountain National Park, which was a habitat for the now-extinct Eastern Elk. The elk being brought into the park are Manitoba elk, which are not as well-adapted to the habitat as the Eastern Elk were. Develop a model for the reintroduction program, and specifically, whether the elk will survive or die out.

Year	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
Reintroduced	25	27					5				
Births	4	2	10	8	10	13	19	19	19	25	19
Death											
Poached				1					1	1	
Sickness	1	5	2	6	2			3	2	3	
Accident			1			1	1	1		2	
Predator	1		1	2	5	1	4		2		
Unknown				3				5	3	2	3
Population	27	51	57	53	56	67	86	96	107	124	140

Figure 1: This is the numeric data for the population growth and the corollary effects on the growth, culminating into the current approximate population of the herd in the GSMNP region.

4. *How Integrated is my Neighborhood (School)?* Using publicly-available data, e.g., http://www.actforchildren.org/site/DocServer/2010_Census_Data_Fact_Sheet_by_Chicago_Community_Area.pdf?docID=1741, develop a way to "score" neighborhoods (or schools) on their degree of racial (or economic) integration. Use this scoring system to identify which neighborhoods (or schools) are closest to an ideal, and to make some recommendations for improving integration or identifying (and potentially remediating) the schools or neighborhoods hardest-hit by segregation.