0 mod n-Sum with Size n Subsets of Sets of Integers, as Arising from a Fictional Character Named Frackkaka

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ABSTRACT (A):

The following paper is concerned with 0 mod n-sum (n natural) subsets of n integers; in particular, it attempts to find the smallest k such that any set of k integers contains such a set. The results are somewhat limited, but establish several bounds on k (the final bound applying to the product of two coprime numbers) and, for n's not divisible by the square of any prime, finds an exact value for k. The problem arose out of the special case with n=6, and deals with this case first (k was found to be 11 in this case).

INTRODUCTION (I):

This problem first came up at Hampshire College Summer Studies in Math XL, probably the idea of junior staff member there named Josh Vekhter ("Chief Silliness Officer"). The original statement of the problem from a HCSSiM homework set is

> Frackkakaka is a scam artist who likes to use dice. In order for her newest scam to work, she needs to roll n dice containing a subset of 6 dice containing a subset of 6 dice $\{d_1, d_2, d_3, d_4, d_5, d_6\}$ such that the face-up numbers on the dice in this set sum to something divisible by 6. In other words, she wants to be able to pick 6 d_i's such that $6|d_1+d_2+d_3+d_4+d_5+d_6$. She thinks her scam will always work if she rolls 35 dice. Can you help her prove it? [Yes] She is also afraid that if she rolls too many dice, her victims will accuse her of mischief. What is the fewest number of dice that she can roll and still be sure that her scam will never fail? Prove it.

The original problem with 35 dice is proved fairly easily by application of the pigeonhole (or pigs-in-holes) principle; the question of the smallest n takes more consideration, but we eventually proved it to be 11.

The generalization of the problem replaces 6 throughout the original statement with k, and asks for the smallest n; that is, "what is the smallest n such that in any set of n k-sided dice there is a subset of size k whose sum is divisible by k?" This is essentially the same problem as "what is the smallest n such that every set of n integers has a subset with length k whose sum is divisible by k?" because the set of n integers can be put into mod k and the 0 in mod k can be considered as the k-valued face of a die – this reformulation is what will be used in the rest of the paper, not the version with dice. The conjectured answer to this problem is $2k$-1, but we were only able (with assistance) to prove this for values of k not divisible by the square of any prime; this is one of the major results. The other major result is that, if the conjecture is true for $k=a$ and $k=b$ and a and b are coprime then it must be true for $k=ab$, reducing the problem to proving the conjecture for all values of k that are powers of primes.

Throughout this paper, a "set" can have repeated elements (so $\{3\}\neq\{3, 3\}$), and so is not a normal set.

BODY (B):

For convenience, let $\sum(S)$ denote the sum of the elements of the set S.

Original Problem with *k*=6 (o):

O1: Proof that n=35 works in the original problem for 6-sided dice:

The dice can belong to at most 6 residue classes in mod 6, and so the pigeonhole principle with dice as pigeons and residue classes as holes guarantees that there will be at least 6 dice, $d_1, d_2, d_3, d_4, d_5, d_6$, all congruent to the same a mod 6, so $d_1+d_2+d_3+d_4+d_5+ d_6 \equiv 6a \equiv 0$ mod 6; so $d_1+d_2+d_3+d_4+d_5+d_6$ is a multiple of 6. In fact, this proof technique would actually work for n=31.

O2: Proof that n=11 works in the original problem for 6-sided dice:

Consider the generalization of the problem with k=3. n=5 works: if a set S of five integers does not contain each mod 3 remainder, then by pigeonhole principle (on 5 integers, $\leq$ 2 residue classes) there will be at least 3 integers in S which have the same remainder mod 3, and so their sum will be 0 mod 3; and if S does contain each remainder mod 3, then there are some *a, b, c* in S with $a \equiv 0 \bmod 3$, $b \equiv 1 \bmod 3$, and $c \equiv 2 \bmod 3$, so that $a+b+c \equiv 0+1+2 \equiv 0 \bmod 3$.

Now consider the k=6 case: take a set of integers S with length 11. 11 > 5, so there is a 3-element subset A_1 of S whose sum is 0 mod 3. $S \setminus A_1$ has size 8, which is larger than five, so there is a 3-element subset A_2 of $S \setminus A_1$ whose sum is 0 mod 3; note that A_2 is a subset of S. $S \setminus A_1 \setminus A_2$ has size 5, so there is a 3-element subset A_3 of $S \setminus A_1 \setminus A_2$ whose sum is 0 mod 3; note that A_3 is a subset of S. By pigeonhole principle (3 ∑'s, 2 remainders mod 2), two of the $\sum(A_i)$'s – $\sum(A_u)$ and $\sum(A_v)$ – must be congruent mod 2, so their sum is 0 mod 2. Because $\sum(A_u)$ and $\sum(A_v)$ are both 0 mod 3, $\sum(A_u)+\sum(A_v) \equiv 0 \bmod 3$, in addition to being 0 mod 2; so $\sum(A_u)+\sum(A_v) \equiv 0 \bmod 6$. But since A_u and A_v are disjoint by their selection, $\sum(A_u)+\sum(A_v)=\sum(A_u \cup A_v)$, and $A_u \cup A_v$ is also a subset of S and has size 6 because $|A_u|=|A_v|=3$; so there is a 6-element subset of S whose sum is 0 mod 6 (it is $A_u \cup A_v$), QED.

<u>O3: Proof that n=11 is the smallest n that works in the original problem for 6-sided dice:</u>

n=10 does not work, because S={0, 0, 0, 0, 0, 1, 1, 1, 1, 1} has no subsets of size 6 with 0 mod 6 sums. Then no $n < 10$ can work, because (proof by contradiction) if an $n < 10$ worked then the above S would contain an n-element subset T with a 6-element subset A with $\sum(A) = 0 \bmod 6$; since A is also a subset of S, S would then contain a 6-element subset with a 0 mod 6 sum, →←. Therefore, n=11 is the smallest n that works in the original problem for 6-sided dice.

Results on the Generalization (g):

Let $f(x)$ be the answer to the generalization's question; that is, $f(x) = \min\{n \in \mathbb{N}: (\forall S \subset \mathbb{N})|S| = n \rightarrow \exists A \subset S: \sum(A) \equiv 0\ mod\ x\ and\ |A| = x\}$. $\{n \in \mathbb{N}: (\forall S \subset \mathbb{N})|S| = n \rightarrow \exists A \subset S: \sum(A) \equiv 0\ mod\ x\ and\ |A| = x\} \subseteq \mathbb{N}$; and $x^2 \in \{n \in \mathbb{N}: (\forall S \subset \mathbb{N})|S| = n \rightarrow \exists A \subset S: \sum(A) \equiv 0\ mod\ x\ and\ |A| = x\}$, because by pigeonhole (x^2 integers, x remainders mod x) there will be at least x integers with the same remainder mod x in any set of x^2 integers, and their sum will be 0 mod x; so $\{n \in \mathbb{N}: (\forall S \subset \mathbb{N})|S| = n \rightarrow \exists A \subset S: \sum(A) \equiv 0 \; mod \; x \; and \; |A| = x\}$ is a nonempty subset of the naturals, so it has a least element, so $f(x)$ is defined for all natural x.

G1: $2a\text{-}1 \leq f(a) \leq a^2$:

The set S containing (a-1) 0's and (a-1) 1's contains no subsets of size a with sums 0 mod a, because sums of a-element subsets of S area always between 1 and a-1, inclusive; therefore $f(a) < 2a-1$ is always false, because for any $n < 2a-1$ it is possible to select an n-element subset of S, which also cannot have any subsets with size a that sum to 0 mod a (because if it did then S would as well).

A set S with a^2 elements is guaranteed by the pigeonhole principle (a^2 elements, a remainders mod a) to have at least a elements that are congruent mod a, and their sum will be 0 mod a.

G2: $(a, b) = 1 \; and \; f(a) = 2a - 1 \; and \; f(b) = 2b - 1 \rightarrow f(ab) = 2ab - 1$

Let S be a set of $2ab$-1 integers. Select a subset K_0 of S with $|K_0|=a-1$; $|S \setminus K_0|=(2ab-1)-(a-1) = a(2b-1)$. Select a subset A_i of $S \setminus K_i$ with $|A_i|=a$, so that $|A_i \cup K_i| = 2a-1$ so there is a subset X_i of $A_i \cup K_i$ with $\sum(X_i) \equiv 0$ mod a and $|X_i|=a$; let $K_{i+1} = A_i \cup K_i \setminus X_i$. This construction gives $2b$-1 pairwise disjoint sets $X_1, X_2 \ldots X_{2b-1}$ with $\sum(X_i) \equiv 0$ mod a and $|X_i|=a$. The set T = $\{\sum(X_i): 1 \leq i \leq 2b-1\}$ has $2b$-1 elements, and so has a subset U with the property that $\sum(U) \equiv 0$ mod b and $|U|=b$; however, each element of U is $\sum(X_i)$, for some i, so $\sum(U) \equiv 0 \; mod \; b \rightarrow \sum_{i: \sum(X_i) \in U} \sum(X_i) \equiv 0 \; mod \; b \rightarrow \sum\left(\bigcup_{i: \sum(X_i) \in U} X_i\right) \equiv 0 \; mod \; b$; by construction, $\sum(X_i) \equiv 0$ mod a for each i, so $\sum\left(\bigcup_{i: \sum(X_i) \in U} X_i\right) \equiv 0 \; mod \; a$. Since $(a, b)=1$, $a \middle| \sum\left(\bigcup_{i: \sum(X_i) \in U} X_i\right) \; and \; b \middle| \sum\left(\bigcup_{i: \sum(X_i) \in U} X_i\right) \rightarrow ab \middle| \sum\left(\bigcup_{i: \sum(X_i) \in U} X_i\right)$; also, since the X_i's are disjoint, $|\bigcup_{i: \sum(X_i) \in U} X_i|=ab$ (there are b sets, each with cardinality a) and $\bigcup_{i: \sum(X_i) \in U} X_i \subset S$, so S has a subset of size ab whose sum is divisible by ab (it is $\bigcup_{i: \sum(X_i) \in U} X_i$). Additionally, $f(ab) \geq 2ab$-1 by G1, so $f(ab)=2ab$-1.

Note: the result of O3 could be obtained by proving that $f(2)=3$ and $f(3)=5$ (as was done in O3) and then applying G2.

This is actually a corollary of a more general theorem:

G3: if $(a, b) = 1$, $f(ab) \leq f(a)+a(f(b)-1)$ and $f(ab) \leq f(b)+b(f(a)-1)$.

Let S be a set of $f(a)+a(f(b)-1)$ integers. Select a subset K_0 of S with $|K_0|=f(a)-a$; $|S \setminus K_0|=(f(a)+a(f(b)-1))-(f(a)-a) = af(b)$. Select a subset A_i of $S \setminus K_i$ with $|A_i|=a$, so that $|A_i \cup K_i| = f(a)$ so there is a subset X_i of $A_i \cup K_i$ with $\sum(X_i) \equiv 0 \bmod a$ and $|X_i|=a$; let $K_{i+1} = A_i \cup K_i \setminus X_i$. This construction gives $f(b)$ pairwise disjoint sets $X_1, X_2 \ldots X_{2b-1}$ with $\sum(X_i) \equiv 0 \bmod a$ and $|X_i|=a$ for each i. The set $T = \{\sum(X_i)\colon 1 \leq i \leq f(b)\}$ has $f(b)$ elements, and so has a subset U with the property that $\sum(U) \equiv 0 \bmod b$ and $|U|=b$; however, each element of U is $\sum(X_i)$, for some i, so $\sum(U) \equiv 0 \; mod \; b \rightarrow \sum_{i:\, \sum(X_i)\in U} \sum(X_i) \equiv 0 \; mod \; b$ and $\sum_{i:\, \sum(X_i)\in U} \sum(X_i) \equiv 0 \; mod \; b \rightarrow \sum\left(\bigcup_{i:\, \sum(X_i)\in U} X_i\right) \equiv 0 \; mod \; b$; by construction, $\sum(X_i) \equiv 0 \bmod a$ for each i, so $\sum\left(\bigcup_{i:\, \sum(X_i)\in U} X_i\right) \equiv 0 \; mod \; a$. Since $(a, b)=1$, $\sum\left(\bigcup_{i:\, \sum(X_i)\in U} X_i\right) \equiv 0 \; mod \; a \; and \; \sum\left(\bigcup_{i:\, \sum(X_i)\in U} X_i\right) \equiv 0 \; mod \; b \rightarrow \sum\left(\bigcup_{i:\, \sum(X_i)\in U} X_i\right) \equiv 0 \; mod \; ab$; also, since the X_i's are disjoint, $|\bigcup_{i:\, \sum(X_i)\in U} X_i| = ab$ (there are b sets, each with cardinality a) and $\bigcup_{i:\, \sum(X_i)\in U} X_i \subset S$, so S has a subset of size ab whose sum is divisible by ab (it is $\bigcup_{i:\, \sum(X_i)\in U} X_i$). $|S|= f(a)+a(f(b)-1)$, so $f(ab) \leq f(a)+a(f(b)-1)$. The proof goes analogously for the other inequality (reverse a and b).

G2 is the result of applying this theorem when $f(a)=2a-1$ and $f(b)=2b-1$, because then G3 says that $f(ab) \leq 2ab-1$ and G1 says that $f(ab) \geq 2ab-1$.

G4: for prime p, $f(p)=2p-1$ (proof strategy suggested by Ramin Takloo-Bighash):

Let $S = \{d_1, d_2 \ldots d_{2p-1}\}$ be a subset of the integers, let $A_1, \; A_2 \ldots A_{\binom{2p-1}{p}}$ be all of the p-element subsets of S, and let $K = \sum(A_1)^{p-1} + \sum(A_2)^{p-1} \ldots \sum(A_{\binom{2p-1}{p}})^{p-1}$. Consider a given product of the form $d_{i_1}^{a_1} d_{i_2}^{a_2} \ldots d_{i_j}^{a_j}$ (with $\sum a_i = p-1$) that appears in K; this product will occur in any $\sum(A_i)^{p-1}$ where $d_{i_1}, d_{i_2} \ldots d_{i_j} \in A_i$, and there are $\binom{2p-j-1}{p-j}$ such A_i's because j d_i's have been selected, and so p-j d_i's are required to complete a set of size p – so $d_{i_1}^{a_1} d_{i_2}^{a_2} \ldots d_{i_j}^{a_j}$ will appear $\binom{2p-j-1}{p-1}$ times in K. $\binom{2p-j-1}{p-j} = \binom{2p-j-1}{p-1} = \frac{(2p-j-1)(2p-j-2)(2p-j-3)\ldots(p-j+1)}{(p-1)!}$, so $(2p-j-1)(2p-j-$

$2)(2p-j-3)\ldots(p-j+1) \equiv \binom{2p-j-1}{p-j}(p-1)!\,mod\ p$, so by Wilson's Theorem $(2p-j-1)(2p-j-2)(2p-j-3)\ldots(p-j+1) \equiv \binom{2p-j-1}{p-j}(-1)mod\ p$. $1 \le j \le p$-1 because it is impossible to multiply fewer than 1 d_i's together and because $j > p$-1 would give a $d_{i_1}^{a_1}d_{i_2}^{a_2}\ldots d_{i_j}^{a_j}$ with $\sum a_i > p$-1; $j \le p-1 \rightarrow 2p-j-1 \ge p$ and $j \ge 1 \rightarrow p-j+1 \le p$, so one of $2p$-j-1, $2p$-j-2…p-j+1 must equal p; therefore $(2p-j-1)(2p-j-2)(2p-j-3)\ldots(p-j+1) \equiv 0\ mod\ p$, because it contains a factor of p. So $(2p-j-1)(2p-j-2)(2p-j-3)\ldots(p-j+1) \equiv \binom{2p-j-1}{p-j}(-1)mod\ p \rightarrow 0 \equiv \binom{2p-j-1}{p-j}(-1)mod\ p \rightarrow \binom{2p-j-1}{p-j} \equiv 0\ mod\ p$. The coefficient of $d_{i_1}^{a_1}d_{i_2}^{a_2}\ldots d_{i_j}^{a_j}$ in each $\sum(A_i)^{p-1}$ will be the same (call it c_i) because the only difference between the A_i's are the d_i's not in $d_{i_1}^{a_1}d_{i_2}^{a_2}\ldots d_{i_j}^{a_j}$, and they will not contribute to the product at all; therefore the coefficient of $d_{i_1}^{a_1}d_{i_2}^{a_2}\ldots d_{i_j}^{a_j}$ in K is $\binom{2p-j-1}{p-j}c_i \equiv (0)c_i \equiv 0\ mod\ p$. This holds for any $d_{i_1}^{a_1}d_{i_2}^{a_2}\ldots d_{i_j}^{a_j}$, and K is the sum of these, so $K \equiv 0\ mod\ p$. Assume towards contradiction that no $\sum(A_i)$ is 0 mod p; then, by Fermat's Little Theorem, $K = \sum(A_1)^{p-1} + \sum(A_2)^{p-1}\ldots\sum\left(A_{\binom{2p-1}{p}}\right)^{p-1} \equiv 1+1+1\ldots1 \equiv \binom{2p-1}{p} \equiv \binom{2p-1}{p-1}\ mod\ p$, so $K \equiv \frac{(2p-1)(2p-2)\ldots(p+1)}{(p-1)!}mod\ p \rightarrow (-1)!\,K \equiv (2p-1)(2p-2)\ldots(p+1) \equiv (p-1)(p-2)\ldots1\ mod\ p \rightarrow (p-1)!\,K \equiv (p-1)!\,mod\ p \rightarrow\ \equiv 1\ mod\ p$; however, it was just proven that K is 0 mod p, which is a contradiction, so therefore some $\sum(A_i)$ is 0 mod p.

G5: $f(n)$=2n-1 for any n not divisible by the square of any prime:

If n is not divisible by the square of any prime, then all exponents in the prime factorization of n must be smaller than 2, so they are 1; that is, $n = p_1p_2p_3\ldots p_k$, for some prime p_i's. Proof proceeds by induction on k: the statement is true for k=1 (this is G4), so the base case holds. Next suppose that the statement holds for k=a; then any n of the form $p_1p_2\ldots p_{a+1}$ is equal to the product of $p_1p_2\ldots p_a$ and p_{a+1}, and $(p_1p_2\ldots p_a, p_{a+1})$=1 because the p_i's are distinct, so by G2 $f(n)$=2n-1.

Note: a program to check every possible set of 7 integers mod 4 (Appendix) found that $f(4) \leq 7$; this and G1 indicate that $f(4) = 7 = 2(4)-1$.

Failed attempted proof of the generalization by reformulation with vectors (f):

The mod a sum of a set A is congruent to $\sum_i (c_i i)$ mod a, where c_i is the number of elements of A congruent to i mod a. In particular, if $|A|=a$, $\sum(A) \equiv \sum c_i i \equiv (c_0, c_1, c_2 \ldots c_{a-1}) \bullet (0, 1, 2 \ldots a-1) \; mod \; a$. Every subset A of the integers can be considered in a vector form $V(A) = (c_0, c_1, c_2 \ldots c_{a-1})$, where each c_i is the number of elements of A congruent to i mod a. $|A|=V(A)\bullet(1, 1\ldots1)$, because this gives the sum of the c_i's (and the residue classes mod x are a partition of A).

For an odd number a and a set S with $f(a)$ elements, let $V(S)=(s_0, s_1, s_2 \ldots s_{a-1})$. Then, for any subset A of S, $V(A)$ must have its i^{th} component be nonnegative and less than or equal to s_i; that is, if $\mathbf{u_i}$ is the vector with i^{th} component 1 and all other components 0, that $0 \leq V(A)\bullet\mathbf{u_i} \leq V(S)\bullet\mathbf{u_i}$. This provides another reformulation: the problem asks for the smallest n such that for every vector $\mathbf{Y}$ with nonnegative integral components satisfying $\mathbf{Y}\bullet(1, 1, 1\ldots1) = n$ ($\mathbf{Y}$ represents $V(S)$), there is a vector $\mathbf{X}$ such that $0 \leq \mathbf{X}\bullet\mathbf{u_i} \leq \mathbf{Y}\bullet\mathbf{u_i}$ for all i, $\mathbf{X}\bullet(1, 1\ldots1)=a$ and $\mathbf{X}\bullet(0, 1, 2\ldots a-1) \equiv 0$ mod a ($\mathbf{X}$ represents $V(A)$); if $0 \leq \mathbf{X}\bullet\mathbf{u_i} \leq \mathbf{Y}\bullet\mathbf{u_i}$, $\mathbf{X}\bullet(1, 1\ldots1)=a$ and $\mathbf{X}\bullet(0, 1, 2\ldots a-1) \equiv 0$ mod a, then $-1 \leq [\mathbf{X}-(1, 1\ldots1)]\bullet\mathbf{u_i} \leq [\mathbf{Y}-(1, 1\ldots1)]\bullet\mathbf{u_i}$, $[\mathbf{X}-(1, 1\ldots1)]\bullet(1, 1\ldots1)=0$ and $[\mathbf{X}-(1, 1\ldots1)]\bullet(0, 1, 2\ldots a-1) \equiv 0$ mod a. $\mathbf{Y}-(1, 1\ldots1)$ has the properties that all of its components are at least -1 and $\mathbf{Y}\bullet(1, 1\ldots1) = n-a$. Let $Z = \{\boldsymbol{X}: \boldsymbol{X} \bullet (1, 1 \ldots 1) = 0 \; and \; \boldsymbol{X} \bullet (0, 1, 2 \ldots a-1) \equiv 0 \; mod \; a\}$. Now the statement that every set of size n contains an a-element subset with size a whose sum is 0 mod a is the same as the statement that for every vector $\mathbf{Y'}$ with components at least -1 satisfying $\mathbf{Y'}\bullet(1, 1\ldots1)=n-a$ there is some $\mathbf{X}$ in Z such that $-1 \leq \mathbf{X}\bullet\mathbf{u_i} \leq \mathbf{Y'}\bullet\mathbf{u_i}$. The conjecture is that $f(a)=2a-1$, and by G1 proving this reformulation would prove the original problem. The reformulation with vectors is potentially useful because Z is a vector space, so it is possible to perform operations on its elements (this couldn't be done in the original statement of the problem because order in sets is unimportant, and so it is unclear what elements should add); however, we were unable to make any progress with this reformulation, as it comes at the expense of being somewhat unwieldy.

CONCLUSION:

The most significant results were results G3 and G5: G3 established bounds on the function f – it found that $(a, b) = 1 \rightarrow f(ab) \leq f(a) + a(f(b) - 1)$ – and results G3 and G4 combine to prove (G5) that $f(n)=2n-1$ for any n not divisible by the square of any prime, which is the other major result.

Further investigation will likely be into sums of numbers of the form $\sum(A_i)^{p^a - p^{a-1}}$, as the p^a-p^{a-1}th power of a number not 0 mod p is always 1 mod p^a; however, as numbers that are 0 mod p^a are not the only numbers that become 0 when raised to p^a-p^{a-1} (numbers 0 mod p will as well), this approach seems less promising for the general p^a case than it was for the general p case. We may make some investigations into the properties of the Z mentioned in B.f; this is a reformulation that at least gives the problem some structure, and involves a set closed under various operations. It was assumed in the above reformulation that a is an odd integer, because all powers of primes other than 2 are odd (and proving the conjecture can be done by proving the conjecture for all powers of primes); the case where $a=2^t$, for some t, would require further investigation.

The author learned nothing particularly universal about research or math, other than apparent intractability of problems involving primes; he was also reminded of a useful characteristic of numbers that are not 0 mod p (their p-1$^{\text{th}}$ powers are 1), and was shown a use of this property in proof.

APPENDIX:

```java
import java.util.Vector;

public class pointless_stuff2_part3
{
	public static void main(String args [])
	{
		int n = 4;
		int k = 2*n-1;
		int [] set = new int [k];
		for (int i = 0; i < k; i++) set [i] = 0;
		Vector<int []>indices = subsets(k, n);
		boolean yes = true, yes2 = false;
		int s = 0;
		for (int i = 0; set [k-1] != n-1 && yes; i++)
		{
			yes2 = false;
			set [0]++;
			for (int j = 0; j < k; j++)
				if (set [j] == n)
				{
					set [j+1]++;
					set [j] = 0;
				}
			for (int j = 0; j < indices.size() && !yes2; j++)
			{
				s = 0;
				for  (int h = 0; h < n; h++)
					s += set [indices.elementAt(j) [h] - 1];
				if ((double)s / n == (int)((double)s / n)) yes2 = true;
			}
			if (!yes2) yes = false;
		}
		if (yes) System.out.print("yes");
		else System.out.print("no");
	}
	static Vector<int []>allsubsets(int n)//gives all subsets of {1, 2...n}
	{
		Vector<int []>subsets = new Vector<int []>();
		if (n > 0)
		{
			Vector<int []>subsets2 = allsubsets(n-1);
			for (int i = 0; i < subsets2.size(); i++)
			{
				subsets.addElement(subsets2.elementAt(i));
				subsets.addElement(new int [subsets2.elementAt(i).length+1]);
				for (int j = 0; j < subsets2.elementAt(i).length; j++)
					subsets.lastElement() [j] = subsets2.elementAt(i) [j];
				subsets.lastElement() [subsets2.elementAt(i).length] = n;
			}
		}
		else subsets.addElement(new int [] {});
		return subsets;
	}
	static Vector<int []>subsets(int n, int k)//gives all subsets of {1, 2...n} with size k
	{
		Vector<int []>subsets = allsubsets(n);
		for (int i = 0; i < subsets.size(); i++)
			if (subsets.elementAt(i).length != k)
			{
				subsets.removeElementAt(i);
				i--;
			}
		return subsets;
	}
}
```