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Patterns in the Period Lengths of Simple Periodic Continued Fractional Representations of Square Roots of Integers Near Perfect Squares

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Abstract

There exist simple periodic continued fractions which converge to the square roots of integers. These fractions have predictable period lengths, k, and terms, $[a_n]$, when the integer itself lies within a distance of two from a perfect square. First, we explore the recursive nature of determining these terms and propose a general method of finding the value of x_1 based on $\sqrt{\beta^2} = a_0 + \frac{1}{x_1}$. Then we more deeply investigate the patterns that form in the period lengths and values of a_n when the square root of an integer lies close to another integer by proving the identities of these values of a_n and confirming their period lengths.

1 Introduction

1.1 Varieties of Continued Fractions

Continued fractions [1] are combinations of two series, $[a_n] = [a_0; a_1, a_2, \ldots]$ and $[b_n] = [b_1, b_2, \ldots]$ where $\forall n$, $a_n, b_n \in \mathbb{Q}$. The most general form of a continued fraction can be written as

$$x = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cdots}}$$

A more specific type of continued fraction is the *simple continued fraction* [2] where $\forall n$, $b_n = 1$:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

Any continued fraction that has a repeating sequence of k_1 values of a_n and a repeating sequence of k_2 values of b_n is referred to as a *periodic continued fraction* [3]. However, when dealing with simple periodic continued fractions, $b_n = 1\ \forall n$ and $k_2 = 1$; k_1 becomes the only relevant value and is referred to as simply k, the period length of $[a_n]$. Note that the period length of any simple periodic continued fraction does not include a_0. This leads to a shorthand method of writing the values of a_n: $[a_n] = [a_0; \overline{a_1, a_2, \cdots, a_k}]$, indicating that those values of a_n eventually repeat as shown below:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_k + \cfrac{1}{a_1 + \cdots}}}}$$

Based on experimental observations in determing the values of a_n for simple continued fractional representations of $\sqrt{\beta^2}$, such fractions are periodic as well. However, in order to prove that such simple periodic continued fractions are equivalent to $\sqrt{\beta^2}$, it first must be shown that continued fractions are convergent.

1.2 Convergence of Continued Fractions

Continued fractions have been studied extensively in the past and the concept of their convergence to a single value is not a new one. Thomas Hawkins [4] notes that Seidel and Stern proved in 1846 and 1848, respectively, "that any continued fraction with positive coefficients converges if and only if at least one of the two series diverges:

$$\sum_{n=1}^{\infty} \frac{b_1 b_3 \cdots b_{2n-1}}{b_2 b_4 \cdots b_{2n}} a_{2n}, \sum_{n=1}^{\infty} \frac{b_2 b_4 \cdots b_{2n}}{b_1 b_3 \cdots b_{2n-1}} a_{2n+1}\text{”}$$

(adjustments made to accomodate differences in notation). When working with simple periodic continued fractions the fact that both of these sequences diverge is almost trivial. As all b_n terms are equal to 1, these summations are reduced to the sums of all even values of a_n and all odd values of a_n, all of which are positive real numbers. Thusly, the sums will never decrease in size and the series will diverge, and the continued fraction itself will converge.

1.3 Period Lengths of Simple Periodic Continued Fractional Representations of $\sqrt{\beta^2}$

While the period lengths of $[a_n]$ vary for values of β^2, there is an interesting correlation between this length and the distance of these values of β^2 from a perfect square. When β^2 is at most a distance of 2 from a perfect square, the period lengths and the values of $[a_n]$ become predictable, the investigation of which is the primary focus of the following sections.

2 Determining $[a_n]$ for $\sqrt{\beta^2}$, $\beta \in \mathbb{R}$

The process of determining each subsequent value of a_n is somewhat of a recursive one.

In order to determine a_0, take the floor of $\sqrt{\beta^2}$. This value will correspond to the square root of the greatest perfect square that is less than or equal to β^2. A *perfect square* is an integer whose square root is also an integer.

Case 1: If $\sqrt{\beta^2} \in \mathbb{Z}$, then $a_0 = \beta$ and $\nexists a_n \ \forall n > 0$.

Case 2: If $\sqrt{\beta^2} \notin \mathbb{Z}$, then $\lfloor \sqrt{\beta^2} \rfloor = a_0$. Let $\sqrt{\beta^2} = a_0 + \frac{1}{x_1}$. Then $\lfloor x_1 \rfloor = a_1$. Now $\sqrt{\beta^2} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{x_2}}$ and $\lfloor x_2 \rfloor = a_2$. This process continues such that for $\sqrt{\beta^2}$ of period length k, $x_1 = x_{k+1}$ and $\lfloor x_n \rfloor = a_n$ $\forall n > 0$.

Appendix A provides the first six values of x_n for $1 \leq \beta^2 \leq 30$ as well as the period lengths of the simple periodic continued fraction representations [5].

2.1 Note of Interest

Brief inspection of Appendix A shows that there are patterns as one travels down the table. One of the more easily distinguishable patterns can be found in the values of x_1 as every non-perfect square has a period length of at least one, whereas not all non-perfect squares have a period length of at least 2. The general method of determining x_1 for $\sqrt{\beta^2}$ is as follows:

$$\sqrt{\beta^2} = a_0 + \frac{1}{x_1}$$

$$\sqrt{\beta^2} = \frac{a_0 x_1 + 1}{x_1}$$

$$\beta^2 = \frac{{a_0}^2 {x_1}^2 + 2a_0 x_1 + 1}{{x_1}^2}$$

$$\beta^2 {x_1}^2 = {a_0}^2 {x_1}^2 + 2a_0 x_1 + 1$$

$$0 = ({a_0}^2 - \beta^2){x_1}^2 + 2a_0 x_1 + 1$$

$$x_1 = \frac{1}{\beta - a_0}$$

$$x_1 = \frac{1}{\beta - a_0} \cdot \frac{\beta + a_0}{\beta + a_0}$$

$$x_1 = \frac{\beta + a_0}{\beta^2 - a_0}$$

3 Patterns Around Perfect Squares

As continued fractions have already been proven to converge to a single value, it is sufficient to show that a given set of values of a_n comprises a solution. Appendix B lists the first six numerical values of a_n for $1 \leq \beta^2 \leq 30$ as well as the period lengths [5].

While some period lengths appear to be random when compared to β^2, others appear to have some sort of direct correlation. Further investigation shows that the period lengths of the square roots of integers near perfect squares (within a distance of 2 from a perfect square, to be exact) are always of the same value. To take this discovery even further, the values of a_n themselves follow a pattern. It is important to recognize that β is an integer in the rest of this section.

3.1 Show that $\beta - 1 + \cfrac{1}{1 + \cfrac{1}{\beta - 2 + \cfrac{1}{1 + \cfrac{1}{2(\beta - 1) + \cfrac{1}{1 + \cdots}}}}}$ converges to $\sqrt{\beta^2 - 2}$.

Proof. Assume that $a_0 = \beta - 1$ and that $a_{4n-3} = 1$, $a_{4n-2} = \beta - 2$, $a_{4n-1} = 1$, $a_{4n} = 2(\beta - 1)$ $\forall n > 0$ [1].

Allow δ to represent a portion of the continued fraction:

$$\delta = 1 + \cfrac{1}{\beta - 2 + \cfrac{1}{1 + \cdots}} \tag{3.1.1}$$

Substitute (3.1.1) into the original simple periodic continued fraction and solve for δ:

$$\sqrt{\beta^2 - 2} = \beta - 1 + \frac{1}{\delta}$$

$$\delta = \frac{\sqrt{\beta^2 - 2} + \beta - 1}{2\beta - 3} \tag{3.1.2}$$

Let ϵ represent part of δ:

$$\epsilon = \beta - 2 + \cfrac{1}{1 + \cfrac{1}{2(\beta - 1) + \cdots}} \tag{3.1.3}$$

[1] The only exception to this generalization is when $\beta^2 \leq 4$.

Substitute (3.1.2) and (3.1.3) into (3.1.1) and solve for ϵ:

$$\frac{\sqrt{\beta^2-2}+\beta-1}{2\beta-3} = \delta = 1+\frac{1}{\epsilon}$$

$$\epsilon = \frac{\sqrt{\beta^2-2}+\beta-2}{2} \tag{3.1.4}$$

Let ζ represent part of ϵ:

$$\zeta = 1+\cfrac{1}{2(\beta-1)+\cfrac{1}{1+\cdots}} \tag{3.1.5}$$

Substitute (3.1.4) and (3.1.5) into (3.1.3) and solve for ζ:

$$\frac{\sqrt{\beta^2-2}+\beta-2}{2} = \epsilon = \beta-2+\frac{1}{\zeta}$$

$$\zeta = \frac{\sqrt{\beta^2-2}+\beta-2}{2\beta-3} \tag{3.1.6}$$

Now let η represent part of ζ:

$$\eta = 2(\beta-1)+\cfrac{1}{1+\cfrac{1}{\beta-2+\cdots}} \tag{3.1.7}$$

Substitute (3.1.6) and (3.1.7) into (3.1.5) and solve for η:

$$\frac{\sqrt{\beta^2-2}+\beta-2}{2\beta-3} = \zeta = 1+\frac{1}{\eta}$$

$$\eta = \sqrt{\beta^2-2}+\beta-1 \tag{3.1.8}$$

The next set of substitutions will complete the chain, so to speak. Substitute (3.1.1) and (3.1.8) into (3.1.7) and solve for δ:

$$\sqrt{\beta^2 - 2} + \beta - 1 = \eta = 2(\beta - 1) + \frac{1}{\delta}$$

$$\delta = \frac{\sqrt{\beta^2 - 2} + \beta - 1}{2\beta - 3} \tag{3.1.9}$$

As this value of δ in (3.1.9) is equivalent to that in (3.1.2), then the cycle will repeat and it has been sufficiently proven that this simple periodic continued fraction converges to $\sqrt{\beta^2 - 2}$. Therefore $a_{4n-3} = 1$, $a_{4n-2} = \beta - 2$, $a_{4n-1} = 1$, $a_{4n} = 2(\beta - 1)$ $\forall n > 0$; $\sqrt{\beta^2 - 2} = [\beta - 1; \overline{1, \beta - 2, 1, 2(\beta - 1)}]$.

□

3.2 Show that $\beta - 1 + \cfrac{1}{1 + \cfrac{1}{2(\beta - 1) + \cfrac{1}{1 + \cdots}}}$ converges to $\sqrt{\beta^2 - 1}$.

Proof. Assume that $a_0 = \beta - 1$ and that $a_{2n-1} = 1$, $a_{2n} = 2(\beta - 1)$ $\forall n > 0$.

The original continued fraction can be simplified by assigning a variable to one of its parts.

$$\delta = 1 + \cfrac{1}{2(\beta - 1) + \cfrac{1}{1 + \cdots}} \tag{3.2.1}$$

Now substitute this value into the simple periodic continued fraction itself:

$$\sqrt{\beta^2 - 1} = \beta - 1 + \frac{1}{\delta} \tag{3.2.2}$$

Solving for δ results in

$$\delta = \frac{\sqrt{\beta^2 - 1} + \beta - 1}{2(\beta - 1)} \tag{3.2.3}$$

Now let ϵ be defined as follows:

$$\epsilon = 2(\beta - 1) + \cfrac{1}{1 + \cfrac{1}{2(\beta - 1) + \cdots}} \tag{3.2.4}$$

$$\epsilon = 2(\beta - 1) + \frac{1}{\delta} \tag{3.2.5}$$

Substitute (3.2.4) into (3.2.1) and solve for ϵ:

$$\delta = 1 + \frac{1}{\epsilon}$$

$$\epsilon = \sqrt{\beta^2 - 1} + b - 1 \tag{3.2.6}$$

To check that the period does, in fact, repeat, substitute (3.2.6) into (3.2.5) and solve for δ:

$$\sqrt{\beta^2 - 1} + b - 1 = 2(\beta - 1) + \frac{1}{\delta}$$

$$\delta = \frac{\sqrt{\beta^2 - 1} + \beta - 1}{2(\beta - 1)} \tag{3.2.7}$$

The value of δ in (3.2.7) is the same as that in (3.2.3). Therefore $a_0 = \beta - 1$ and $a_{2n-1} = 1$, $a_{2n} = 2(\beta - 1)\ \forall n > 0$; $\sqrt{\beta^2 - 1} = [\beta - 1; \overline{1, 2(\beta - 1)}]$.

□

3.3 Show that β converges to $\sqrt{\beta^2}$.

By definition $\beta = \sqrt{\beta^2}$. When β is subtracted from $\sqrt{\beta^2}$ there is no remainder from which further elements of the sequence $\{a_n\}$ can be derived. Therefore $a_0 = \beta$; $\sqrt{\beta^2} = [\beta]$.

3.4 Show that $\beta + \cfrac{1}{2\beta + \cfrac{1}{2\beta + \cdots}}$ converges to $\sqrt{\beta^2+1}$.

Proof. Assume that $a_0 = \beta$ and $a_n = 2\beta \; \forall n > 0$.

$$\sqrt{\beta^2+1} = \beta + \cfrac{1}{2\beta + \cfrac{1}{2\beta + \cdots}} \tag{3.4.1}$$

In order to simplify (3.4.1) further, a more manipulable variable, δ can be introduced:

$$\delta = 2\beta + \frac{1}{2\beta + \cdots} \tag{3.4.2}$$

Substitute (3.4.2) into (3.4.1):

$$\sqrt{\beta^2+1} = \beta + \frac{1}{\delta}$$

$$\sqrt{\beta^2+1} = \frac{\delta\beta + 1}{\delta}$$

$$\beta^2 + 1 = \frac{\delta^2\beta^2 + 2\delta\beta + 1}{\delta^2} \tag{3.4.3}$$

Because δ begins to repeat itself another substitution can be made to reduce (3.4.2). This reduction proves to be useful in simplifying (3.4.3).

$$\delta = 2\beta + \frac{1}{\delta}$$

$$\delta = \frac{2\beta\delta + 1}{\delta}$$

$$\delta^2 = 2\beta\delta + 1$$

$$\delta^2 = 2\beta\delta + 1 \tag{3.4.4}$$

Substitute (3.4.4) into (3.4.3) and simplify:

$$\beta^2 + 1 = \frac{\beta^2(2\beta\delta + 1) + 2\beta\delta + 1}{2\beta\delta + 1} \tag{3.4.5}$$

$$\begin{aligned} (\beta^2 + 1)(2\beta\delta + 1) &= \beta^2(2\beta\delta + 1) + 2\delta\beta + 1 \\ \beta^2(2\beta)\delta + \beta^2 + 2\beta\delta + 1 &= \beta^2(2\beta)\delta + \beta^2 + 2\delta\beta + 1 \end{aligned} \tag{3.4.6}$$

From here it becomes apparent that the expressions on both sides of the equation are equal. Therefore $a_0 = \beta$ and $a_n = 2\beta \; \forall n > 0$; $\sqrt{\beta^2 + 1} = [\beta; \overline{2\beta}]$. □

3.5 Show that $\beta + \cfrac{1}{\beta + \cfrac{1}{2\beta + \cfrac{1}{\beta + \cdots}}}$ converges to $\sqrt{\beta^2 + 2}$.

Proof. Assume that $a_0 = \beta$ and $a_{2n-1} = \beta$, $a_{2n} = 2\beta \; \forall n > 0$.

$$\sqrt{\alpha + 2} = \beta + \cfrac{1}{\beta + \cfrac{1}{2\beta + \cfrac{1}{\beta + \cdots}}} \tag{3.5.1}$$

As in previous proofs, the addition of a variable can simplify the continued fraction into a finite number of terms.

$$\delta = \beta + \cfrac{1}{2\beta + \cfrac{1}{\beta + \cdots}} \tag{3.5.2}$$

Substitute (3.5.2) into (3.5.1):

$$\sqrt{\beta^2 + 2} = \beta + \frac{1}{\delta}$$

$$\sqrt{\beta^2 + 2} = \frac{\delta\beta + 1}{\delta}$$

$$\beta^2 + 2 = \frac{\delta^2\beta^2 + 2\delta\beta + 1}{\delta^2} \tag{3.5.3}$$

However, δ itself can be adjusted so as to have a finite number of terms.

$$\delta = \beta + \cfrac{1}{2\beta + \cfrac{1}{\beta + \cdots}}$$

$$\delta = \beta + \cfrac{1}{2\beta + \cfrac{1}{\delta}} \tag{3.5.4}$$

Manipulate the equation so as not to include any fractional components and equal 0:

$$\delta = \beta + \cfrac{1}{2\beta + \cfrac{1}{\delta}}$$

$$\delta = \beta + \cfrac{1}{\cfrac{2\beta\delta + 1}{\delta}}$$

$$\delta = \beta + \frac{\delta}{2\beta\delta + 1}$$

$$\delta = \frac{2\delta\beta^2 + \beta + \delta}{2\beta\delta + 1}$$

$$2\delta^2\beta + \delta = 2\delta\beta^2 + \beta + \delta$$

$$2\delta^2\beta = 2\delta\beta^2 + \beta$$

$$2\delta^2 = 2\beta\delta + 1$$

$$2\delta^2 - 2\beta\delta - 1 = 0 \tag{3.5.5}$$

Substitute the coefficients of (3.5.5) into the quadratic equation to solve for the positive value of δ:

$$\delta = \frac{\sqrt{\beta^2+2}+\beta}{2} \tag{3.5.6}$$

Substitute (3.5.6) into (3.5.3):

$$\beta^2+2 = \frac{{\frac{\sqrt{\beta^2+2}+\beta}{2}}^2 \beta^2 + 2\frac{\sqrt{\beta^2+2}+\beta}{2}\beta + 1}{{\frac{\sqrt{\beta^2+2}+\beta}{2}}^2} \tag{3.5.7}$$

Expand all terms and simplify wherever possible:

$$\beta^2+2 = \frac{\frac{2\beta^2+2\sqrt{\beta^2+2}\beta+2}{4}\beta^2 + (\sqrt{\beta^2+2}+\beta)\beta + 1}{\frac{2\beta^2+2\sqrt{\beta^2+2}\beta+2}{4}}$$

$$\frac{\beta^4+\sqrt{\beta^2+2}\beta^3+\beta^2}{2} + \beta^2 + \sqrt{\beta^2+2}\beta + 1 = \frac{\beta^4+\sqrt{\beta^2+2}\beta^3+\beta^2}{2} + \beta\sqrt{\beta^2+2} + \beta^2 + 1 \tag{3.5.8}$$

From here it becomes apparent that the expressions on either side of (3.5.8) are equal. Therefore $a_0 = \beta$ and $a_{2n-1} = \beta$, $a_{2n} = 2\beta \ \forall n > 0$; $\sqrt{\beta^2+2} = [\beta; \overline{\beta, 2\beta}]$. □

4 Conclusion

When the square root of an integer is represented by a simple continued fraction, either such a fraction is periodic or the square root itself is an integer. Determining the values of a_n was primarily made possible due to the eventual convergence of simple periodic continued fractions to a single value, but proving the identities of those values of a_n was a direct consequence of noticing that an interesting sequence of numbers kept popping up in the table of period lengths. This sequence– 4, 2, 0, 1, 2– is centered around $\sqrt{\beta^2}$ with a period length of 0 and ranges from $\sqrt{\beta^2-2}$ with a period length of 4 to

$\sqrt{\beta^2+2}$ with a period length of 2. A final table of conclusions can be found on the next page.

Though the math itself that went into proving the values of a_n and the period lengths for the square roots of integers within a distance of two from a perfect square was not all that difficult, it did require a lot of patience. This mostly involved messing around with numbers and equations using a calculator (and then WolframAlpha when equations became more tedious and complex) and seeing what stood out. In this case, the period lengths at first seemed completely random but eventually ideas just clicked into place, usually as the result of thinking, "Could this work again in a slightly different situation?" Everything in mathematics is connected somehow, it just takes a little bit of fiddling to figure it out.

Though not proved explicitly anywhere in this text, Appendix A does have several curious patterns besides the predictable values of x_1. It is possible that one could determine a general solution for x_n, and from this point it could become possible to create some sort of method to identify the n^{th} term of $\sqrt{\beta^2}$ for any β^2. It might also be worthy of study to further investigate the relationship between β^2 and the period length of the continued fraction; beyond anticipated values around perfect squares, these values appear rather random but could possibly be linked to another mathematical topic.

It would also be interesting to investigate a potential link between cube roots and perfect cubes (and for fourth roots and perfect fourths, and so on). Because there exists a predictable period length (as well as values of a_n) for the square roots of integers within a distance of 2 from a perfect square, it could also be the case that there exists a predictable period length for the n^{th} roots of integers within a distance of n from an integer of the form β^n where $\beta \in \mathbb{Z}$.

Let $\beta \in \mathbb{Z}$:

$\forall \beta^2 > 4, \sqrt{\beta^2 - 2}:$ $a_0 = \beta - 1$; $a_{4n-3} = 1 \quad \forall n > 0$; $a_{4n-2} = \beta - 2 \quad \forall n > 0$; $a_{4n-1} = 1 \quad \forall n > 0$; $a_{4n} = 2(\beta - 1) \quad \forall n > 0$	$\beta - 1 + \cfrac{1}{1 + \cfrac{1}{\beta - 2 + \cfrac{1}{1 + \cfrac{1}{2(\beta - 1) + \cfrac{1}{1 + \cdots}}}}}$
$\sqrt{\beta^2 - 1}:$ $a_0 = \beta - 1$; $a_{2n-1} = 1 \quad \forall n > 0$; $a_{2n} = 2(\beta - 1) \quad \forall n > 0$	$\beta - 1 + \cfrac{1}{1 + \cfrac{1}{2(\beta - 1) + \cfrac{1}{1 + \cdots}}}$
$\sqrt{\beta^2}:$ $a_0 = \beta$; $a_n = - \quad \forall n > 0$	β
$\sqrt{\beta^2 + 1}:$ $a_0 = \beta$; $a_n = 2\beta \quad \forall n > 0$	$\beta + \cfrac{1}{2\beta + \cfrac{1}{2\beta + \cdots}}$
$\sqrt{\beta^2 + 2}:$ $a_0 = \beta$; $a_{2n-1} = \beta \quad \forall n > 0$; $a_{2n} = 2\beta \quad \forall n > 0$	$\beta + \cfrac{1}{\beta + \cfrac{1}{2\beta + \cfrac{1}{\beta + \cdots}}}$

A x_n of $\sqrt{\beta^2}$ for $1 \leq \beta^2 \leq 30$

β^2	x_1	x_2	x_3	x_4	x_5	x_6	Period Length
1	-	-	-	-	-	-	0
2	$1+\sqrt{2}$	$1+\sqrt{2}$	$1+\sqrt{2}$	$1+\sqrt{2}$	$1+\sqrt{2}$	$1+\sqrt{2}$	1
3	$\frac{1+\sqrt{3}}{2}$	$1+\sqrt{3}$	$\frac{1+\sqrt{3}}{2}$	$1+\sqrt{3}$	$\frac{1+\sqrt{3}}{2}$	$1+\sqrt{3}$	2
4	-	-	-	-	-	-	0
5	$2+\sqrt{5}$	$2+\sqrt{5}$	$2+\sqrt{5}$	$2+\sqrt{5}$	$2+\sqrt{5}$	$2+\sqrt{5}$	1
6	$\frac{2+\sqrt{6}}{2}$	$2+\sqrt{6}$	$\frac{2+\sqrt{6}}{2}$	$2+\sqrt{6}$	$\frac{2+\sqrt{6}}{2}$	$2+\sqrt{6}$	2
7	$\frac{2+\sqrt{7}}{3}$	$\frac{1+\sqrt{7}}{2}$	$\frac{1+\sqrt{7}}{3}$	$2+\sqrt{7}$	$\frac{2+\sqrt{7}}{3}$	$\frac{1+\sqrt{7}}{2}$	4
8	$\frac{2+\sqrt{8}}{4}$	$2+\sqrt{8}$	$\frac{2+\sqrt{8}}{4}$	$2+\sqrt{8}$	$\frac{2+\sqrt{8}}{4}$	$2+\sqrt{8}$	2
9	-	-	-	-	-	-	0
10	$3+\sqrt{10}$	$3+\sqrt{10}$	$3+\sqrt{10}$	$3+\sqrt{10}$	$3+\sqrt{10}$	$3+\sqrt{10}$	1
11	$\frac{3+\sqrt{11}}{2}$	$3+\sqrt{11}$	$\frac{3+\sqrt{11}}{2}$	$3+\sqrt{11}$	$\frac{3+\sqrt{11}}{2}$	$3+\sqrt{11}$	2
12	$\frac{3+\sqrt{12}}{3}$	$3+\sqrt{12}$	$\frac{3+\sqrt{12}}{3}$	$3+\sqrt{12}$	$\frac{3+\sqrt{12}}{3}$	$3+\sqrt{12}$	2
13	$\frac{3+\sqrt{13}}{4}$	$\frac{1+\sqrt{13}}{3}$	$\frac{2+\sqrt{13}}{3}$	$\frac{1+\sqrt{13}}{4}$	$3+\sqrt{13}$	$\frac{3+\sqrt{13}}{4}$	5
14	$\frac{3+\sqrt{14}}{5}$	$\frac{2+\sqrt{14}}{2}$	$\frac{2+\sqrt{14}}{5}$	$3+\sqrt{14}$	$\frac{3+\sqrt{14}}{5}$	$\frac{2+\sqrt{14}}{2}$	4
15	$\frac{3+\sqrt{15}}{6}$	$3+\sqrt{15}$	$\frac{3+\sqrt{15}}{6}$	$3+\sqrt{15}$	$\frac{3+\sqrt{15}}{6}$	$3+\sqrt{15}$	2
16	-	-	-	-	-	-	0
17	$4+\sqrt{17}$	$4+\sqrt{17}$	$4+\sqrt{17}$	$4+\sqrt{17}$	$4+\sqrt{17}$	$4+\sqrt{17}$	1
18	$\frac{4+\sqrt{18}}{2}$	$4+\sqrt{18}$	$\frac{4+\sqrt{18}}{2}$	$4+\sqrt{18}$	$\frac{4+\sqrt{18}}{2}$	$4+\sqrt{18}$	2
19	$\frac{4+\sqrt{19}}{3}$	$\frac{2+\sqrt{19}}{5}$	$\frac{3+\sqrt{19}}{2}$	$\frac{3+\sqrt{19}}{5}$	$\frac{2+\sqrt{19}}{3}$	$4+\sqrt{19}$	6
20	$\frac{4+\sqrt{20}}{4}$	$4+\sqrt{20}$	$\frac{4+\sqrt{20}}{4}$	$4+\sqrt{20}$	$\frac{4+\sqrt{20}}{4}$	$4+\sqrt{20}$	2
21	$\frac{4+\sqrt{21}}{5}$	$\frac{1+\sqrt{21}}{4}$	$\frac{3+\sqrt{21}}{3}$	$\frac{3+\sqrt{21}}{4}$	$\frac{1+\sqrt{21}}{5}$	$4+\sqrt{21}$	6
22	$\frac{4+\sqrt{22}}{6}$	$\frac{2+\sqrt{22}}{3}$	$\frac{4+\sqrt{22}}{2}$	$\frac{4+\sqrt{22}}{3}$	$\frac{2+\sqrt{22}}{6}$	$4+\sqrt{22}$	6
23	$\frac{4+\sqrt{23}}{7}$	$\frac{3+\sqrt{23}}{2}$	$\frac{3+\sqrt{23}}{7}$	$4+\sqrt{23}$	$\frac{4+\sqrt{23}}{7}$	$\frac{3+\sqrt{23}}{2}$	4
24	$\frac{4+\sqrt{24}}{8}$	$4+\sqrt{24}$	$\frac{4+\sqrt{24}}{8}$	$4+\sqrt{24}$	$\frac{4+\sqrt{24}}{8}$	$4+\sqrt{24}$	2
25	-	-	-	-	-	-	0
26	$5+\sqrt{26}$	$5+\sqrt{26}$	$5+\sqrt{26}$	$5+\sqrt{26}$	$5+\sqrt{26}$	$5+\sqrt{26}$	1
27	$\frac{5+\sqrt{27}}{2}$	$5+\sqrt{27}$	$\frac{5+\sqrt{27}}{2}$	$5+\sqrt{27}$	$\frac{5+\sqrt{27}}{2}$	$5+\sqrt{27}$	2
28	$\frac{5+\sqrt{28}}{3}$	$\frac{4+\sqrt{28}}{4}$	$\frac{4+\sqrt{28}}{3}$	$5+\sqrt{28}$	$\frac{5+\sqrt{28}}{3}$	$\frac{4+\sqrt{28}}{4}$	4
29	$\frac{5+\sqrt{29}}{4}$	$\frac{3+\sqrt{29}}{5}$	$\frac{2+\sqrt{29}}{5}$	$\frac{3+\sqrt{29}}{4}$	$5+\sqrt{29}$	$\frac{5+\sqrt{29}}{4}$	5
30	$\frac{5+\sqrt{30}}{5}$	$5+\sqrt{30}$	$\frac{5+\sqrt{30}}{5}$	$5+\sqrt{30}$	$\frac{5+\sqrt{30}}{5}$	$5+\sqrt{30}$	2

B $\sqrt{\beta^2} = [a_0; a_1, a_2, \ldots, a_n]$, $1 \leq \beta^2 \leq 30$

β^2	a_0	a_1	a_2	a_3	a_4	a_5	a_6	Period Length
1	1	-	-	-	-	-	-	0
2	1	2	2	2	2	2	2	1
3	1	1	2	1	2	1	2	2
4	2	-	-	-	-	-	-	0
5	2	4	4	4	4	4	4	1
6	2	2	4	2	4	2	4	2
7	2	1	1	1	4	1	1	4
8	2	1	4	1	4	1	4	2
9	3	-	-	-	-	-	-	0
10	3	6	6	6	6	6	6	1
11	3	3	6	3	6	3	6	2
12	3	2	6	2	6	2	6	2
13	3	1	1	1	1	6	1	5
14	3	1	2	1	6	1	2	4
15	3	1	6	1	6	1	6	2
16	4	-	-	-	-	-	-	0
17	4	8	8	8	8	8	8	1
18	4	4	8	4	8	4	8	2
19	4	2	1	3	1	2	8	6
20	4	2	8	2	8	2	8	2
21	4	1	1	2	1	1	8	6
22	4	1	2	4	2	1	8	6
23	4	1	3	1	8	1	3	4
24	4	1	8	1	8	1	8	2
25	5	-	-	-	-	-	-	0
26	5	10	10	10	10	10	10	1
27	5	5	10	5	10	5	10	2
28	5	3	2	3	10	3	2	4
29	5	2	1	1	2	10	2	5
30	5	2	10	2	10	2	10	2

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