Abstract:

The Divisor Game is seemingly simple two-person game; but, like so much of math, upon further investigation, it delights one with a plethora of astounding and fascinating patterns. By examining different patterns within the game, we can develop guaranteed winning strategies and make connections to other mathematical subjects.

Paper:

The Divisor Game is seemingly simple two-person game; but, like so much of math, upon further investigation, it delights one with a plethora of astounding and fascinating patterns. The game connects the mathematical subjects of number theory, combinatorics, algebra, and game theory. I discovered the basic Divisor Game simply described on a Julia Robinson Festival website (http://www.msri.org/web/msri/static-pages/-/node/211). To play, one must write down any positive number N, and write down all of its divisors, including 1 and the number itself. Here is a sample board of $N = 15$:

1 3 5 15

To start the game, players take turns crossing out an open divisor of N, and all of the open divisors of that divisor. Open is defined by not already crossed out. The object of the game is to force the other player to cross out N itself. To further clarify, here is an example of a game using $N = 15$. I will show the board after each player's turn, with the numbers that the first player crossed out in red and the second player green. Here is the starting board and all the moves after it:

1	3	5	15
~~1~~	3	5	15
~~1~~	~~3~~	5	15
~~1~~	~~3~~	~~5~~	15
~~1~~	~~3~~	~~5~~	~~15~~

Since the second player is forced to cross out 15 at the end, he/she loses the game, and the first player wins. Is this a random outcome, or is there some pattern within the game that players can use to help them win? Assuming that each player knows the best strategy and is "perfect", in other words, does not make mistakes in playing the game, can we predict the winner of a certain *N*? In this paper, I will show that, by examining different types of *N*'s, we can identify patterns within the game, and create corresponding winning strategies for several types of *N*.

In order to identify our first pattern and corresponding strategy, we will examine the game when *N* is of the form A^x . A is defined as a positive prime number. We will start by writing out the board for an infinite A^x. The board will look like this:

1	A	A^2	A^3	…	A^{x-1}	A^x

Now, it is the first player's move. If the first player crosses out A^{x-1}, then they will also cross out A^{x-2}, A^{x-3}, …, A^2, A^1, and 1, since those are all factors of A^{x-1}. In other words, they will cross out every number on the board except *N* itself. That forces the second player to cross out *N*, leading to the first player being victorious. This pattern and winning strategy will apply to all cases of N = A^x.

Now that we have shown that there is a pattern and corresponding winning strategy for numbers of the form A^x, we will look for similar patterns and winning strategies for prime numbers and products of distinct prime numbers. Let's start with tackling a single prime *N*. Here is the board:

1 A

The winning strategy to this *N* is obvious! If the first player crosses out 1, then the second player is forced to cross out *N* itself, and that leads to the first player winning.

Are there similar patterns and strategies for *N* as the product of two primes? Or three? We will start with the product of two distinct primes, an *N* of the form A^1B^1. A and B are again defined as positive prime numbers. In experimenting with the game when *N* is in the form A^1B^1, I found that a different, diamond-shaped game board highlighted the patterns and winning strategies better. So, I will also redraw the board as a diamond shape so that it has the same divisors but in a different pattern. On each level of the diamond, we put terms made up of zero primes, one prime, two primes, etcetera. Here is the board for A^1B^1:

	AB	
A		B
	1	

If the first player crosses out 1, then there are two options of divisors for the second player to cross out: A and B. If the second player crosses out A, then the first player will cross out B, and the second player will cross out *N*, resulting in the first player's victory. Similarly, if the second player crosses out B, then the first player will cross out A, forcing the second player to cross out *N*. Thus, this game will be over in favor of the first player again. Clearly, in this situation, it is a good strategy for the first player to cross out 1 to begin with; then the game is virtually over. Furthermore, we have figured out that in this two-prime game, the first player will always come out with the win.

Now, we can tackle the situation of the product of three primes with the same diamond-shaped board that we used in looking at AB. Will player one be able to have a similar winning strategy yet again? Here is the board for $A^1B^1C^1$. Like A and B, C is defined as a prime number.

	ABC	
AB	AC	BC
A	B	C
	1	

We have already demonstrated how to win with an AB board. Now, if only the first player can get back to the AB board pattern, they already have a strategy of how to win from there. Say that the first player starts with crossing out the 1. The second player has two choices: first, to cross out a divisor made up of a single prime, such as A, B, or C, or a divisor made up of two distinct primes, AB, AC, or BC. If the second player crosses out a divisor made up of two primes, then we are left with a board of the same pattern as the AB board, and we know the first player can guarantee the win from there, as proven above. But, if the second player chooses to cross out a divisor made up of one prime, then the strategy gets a little harder. Say that the second player crosses out B. Here is what the board would look like:

	ABC	
AB	AC	BC
A	~~B~~	C
	~~1~~	

Now, the first player is going to make a big move. He/she is going to cross out AC, along with its divisors, A and C. Then the board will look like this:

	ABC	
AB	~~AC~~	BC
~~A~~	~~B~~	~~C~~
	~~1~~	

This board now looks like the AB board after the first player has made his/her first move of crossing out the 1. We have already proven above that this is a winning board. Hence, the first player wins again. In other words, by crossing out the 1 first in the ABC board, the first player can guarantee a return to the AB board pattern, and thus a winning strategy.

We have seen a similar pattern and winning strategy for a game in which *N* is made up of a single prime, two primes, or three primes. Let us look more deeply into this pattern to see if we can generalize it. To explore this pattern in more detail, first we will rewrite the AB and ABC boards, this time highlighting the number of terms in each level of the diamond pattern. We will call the bottom level the 0^{th} level, the next-up level the 1^{st} level, and so on and so forth. Here are the AB and ABC boards with levels:

AB board:

2^{nd} level:		AB		(1 term)
1^{st} level:	A		B	(2 terms)
0^{th} level:		1		(1 term)

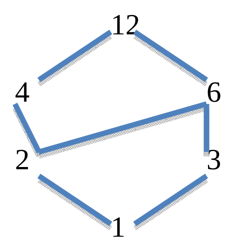
ABC board:

3^{rd} level:		ABC		(1 term)
2^{nd} level:	AB	AC	BC	(3 terms)
1^{st} level:	A	B	C	(3 terms)
0^{th} level:		1		(1 terms)

In the AB board, we can see that on the 0^{th} level there is one term, on the 1^{st} there are two, and on the 2^{nd} there is one. In the ABC board, the pattern of 0^{th} to 3^{rd} level terms is 1-3-3-1. Quite interestingly, these patterns, 1-2-1 and 1-3-3-1, are also the numbers in the rows of Pascal's Triangle.

This is fascinating! Why would the pattern of Pascal's Triangle appear in the pattern of the Divisor Game? The answer is embedded in combinations. To explain, let's look with more detail at the terms on each level of the board. Take the example of the ABC board. There is a total of three different primes that we can make numbers out of: A, B, and C. Take the 1^{st} level of the ABC board. To compute how many terms there are on that level, we can simply compute C(3, 1). This is the combination that we need because, in this case, we have three primes to choose from, and we only need one of them to make a term in this level. So, there are three terms in this level. Similarly, there are C(3, 0) = 1 term in the 0^{th} level, C(3, 2) = 3 terms in the 2^{nd} level, and there are C(3, 3) = 1 term in the 3^{rd} level. As we know, the Pascal's Triangle is made up of combinations. These same combinations are found in the Pascal's Triangle, hence the overlap with the pattern of our Divisor Game. Based on our observation of this pattern, we can generalize the strategy that we use in the ABC board (to get back to the AB board), and say that the winning strategy is simply to get back to one of the distinct prime boards that we know, which now can be categorized as Pascal's Triangle patterns (1-2-1, 1-3-3-1, 1-4-6-4-1, etcetera).

Now, that we have explored the patterns and strategies when *N* is a product of distinct primes, we will finally explore what happens when *N* is in the form A^xB^1. With experimentation, I found that the best way to explore the patterns of *N* in this case is to write the board in yet a different way. This way I will call a web. Here is what the board for A^xB^1will look like, using the example of $2^2 3^1 = 12$:



The lines connect a number with its divisors below, such as 4 connecting to 2, and 12 connecting to 4 and 6.

The best strategy in the 12-web, as in most of the cases we have worked through previously, is for the first player to cross out 1. After the first player crosses out 1, the second player may choose the 4 to cross out, in which case the first player crosses out the 6 and wins. If the second player crosses out the 6, the first player will cross out 4 and win. If the second player crosses out the 2 or the 3, then the first player is presented with a board similar to the AB board, a winning board for the first player.

The pattern in the 12-web can be generally applied algebraically to a web of A^xB^1. The generalized A^xB^1 board will be of the same format as the 12 web; with pure powers of A on the left side of the web, and powers of A times B^1 on the right side of the web. The same strategy as the 12-web applies to the A^xB^1 board, too; in these webs it is also a good strategy for the first player to start by crossing out the 1.

In this paper, I discussed the interesting patterns and corresponding winning strategies of the Divisor Game. I observed patterns with *N*'s of the form A^x, A^1, A^1B^1, $A^1B^1C^1$, and A^xB^1.

Identifying these patterns, and making further connections to the Pascal's Triangle, I was able to develop a few winning strategies for the Divisor Game. Further exploration in this problem could involve figuring out a formal proof of the patterns that I explored. It would also be interesting to examine patterns when *N* is of the form A^xB^y, or even $A^xB^yC^{z\cdot}$ going on forever, with an infinite number of bases with their exponents. Player one certainly always wins with the patterns I explored; perhaps one can even prove that player one simply always wins! As usual in math, the Divisor Game is rich with unexpected nooks and crannies to be explored. The beauty of math shines brightly in this problem.