The Graph that Keeps on Gifting

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Abstract

This paper attempts to find the total number of ways to distribute pairs of gifts in baskets to couples of guests such that no couple has two of the same gift. It then attempts to generalize the problem for n guests, finding (1) whether a distribution exits and (2) how many such distributions exist.

1 Introduction

1.1 Intro to Graph Theory

A **graph** is a collection of **vertices** and **edges** connecting pairs of vertices. The set of vertices is denoted $V(G)$, the set of edges $E(G)$, and thus the graph $G = (V, E)$. In this paper, We will only be considering **simple graphs**, or graphs where there cannot be two edges between the same pair of vertices (**parallel edges**), and an edge cannot connect a vertex to itself (**loop**).

A **proper edge coloring** of a graph is a function linking $E(G)$ to C, where C is a set of distinct colors, such that no two edges that meet at a vertex (adjacent edges) are assigned the same color.

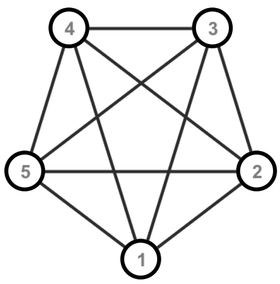
Much of the focus of this paper will be on **complete graphs**, denoted K_n, simple graphs in which every pair of distinct vertices is connected by a unique edge.

1.2 Statement of the Problem

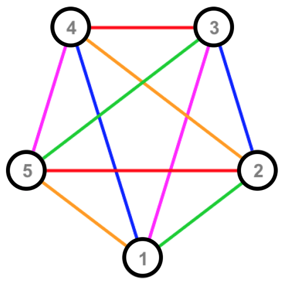
A socialite hosts five couples at his house for a party. He has several copies of five types of gifts, and noting that $10 = \binom{5}{2}$, he decides to assemble one gift basket for each possible pair of gifts and to give one basket to each guest. Seeking to avoid redundancy, he wishes to distribute the baskets in such a way that no couple receives two copies of the same gift between their two baskets. Is this possible? If so, how many ways can it be done? (adapted from a problem set from the Hampshire College Summer Studies in Mathematics).

1.3 Solution

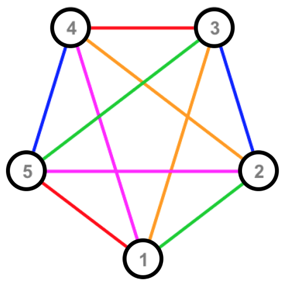
The problem can be visualized and approached through graph theory. Let vertices $v \in V$ represent the five gifts. Let edges $e \in E$ represent the possible baskets, e.g. the edge from v_1 to v_2 represents the basket containing gift 1 and gift 2. Together, the vertex and edge sets V and E, respectively, form a complete graph on 5 vertices, K_5. In this case, the each pair of distinct gifts are connected by a unique basket. The graph of this is below:



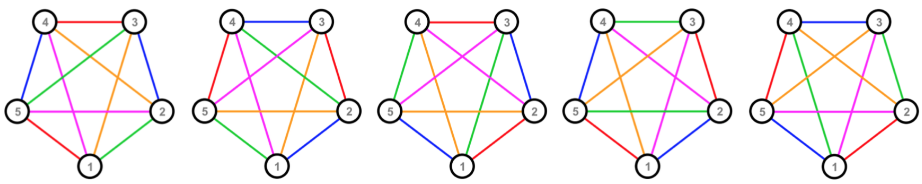
Now, we need to assign guests to gifts in a way such that no two guests in a couple receive a gift in common. A way of doing this is coloring the edges of the graph. Let colors $c \in C$ represent couples. Since each couple must have two guests that each receive a gift, and no couple receives a gift in common, a valid edge coloring of K_5 has 5 colors, each of which assigned to two edges such that no two edges of the same color meet at a vertex. An example of a valid edge coloring for K_5 is:



So this particular distribution of gifts is possible. Onto the next question: How many different ways can the gifts be distributed? The graph above is symmetric; therefore rotating the edges around the vertices produces no change in the coloring. Therefore, let the graph above count as 1 valid edge coloring. However, another coloring exists:



Unlike the coloring before it, this valid coloring is not symmetric. An easy way to notice this is to observe the two green parallel edges; as the graph is rotated while keeping the vertex numberings static, different vertices get the green parallel edges. Thus, this graph adds 5 valid colorings, one for each rotation:



$5 + 1$ valid colorings gives a grand total of 6 valid colorings. Within each coloring, each couple is can be assigned a color $5!$ ways. Additionally, within a couple, guests can switch baskets, increasing the number of ways by a factor of 2^5. Therefore the total number of ways to distribute gifts is $6 \times 5! \times 2^5 = 23040$

2 Generalizing If Distribution is Possible

This section of the paper generalizes if distribution is possible with n gifts. The same simplifying assumption will be maintained, that the number of guests, g = $\binom{n}{2}$, which guarantees that there will be the same number of guests and possible baskets.

2.1 Valid Numbers of Gifts

For the $\binom{n}{2}$ guests to be in couples, $\binom{n}{2}$ must be even.

$$\text{Therefore } \binom{n}{2} = \frac{n(n-1)}{2} = 2m, m \in \mathbb{Z}^+$$
$$\implies n(n-1) = 4m$$
$$\implies 4|n \text{ or } 4|(n-1)$$

$\therefore$ the set of valid possible gifts is the set N = $\{n|$n=4k *or* n=4k+1, $k \in \mathbb{Z}^+$ $\}$

2.2 Determining if a Valid Distribution Exists

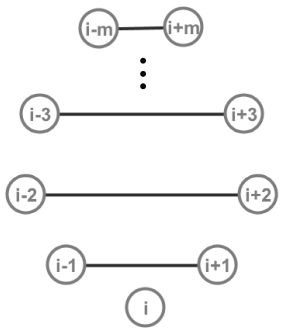
Definition: $C(K_n)$ is valid coloring of the edges of K_n with $\frac{n(n-1)}{4}$ colors such that each color c_1, c_2,...,$c_{\frac{n(n-1)}{4}}$ is assigned to 2 edges, and no 2 edges of the same color meet at a vertex. The actual color assigned to the pairs of edges makes no difference, for example, if the pair assigned color i and the pair assigned color j switched colors, this would not create a new $C(K_n)$ since the pairs no not change; therefore $C(K_n)$ can be thought of as a partitioning of the graph into pairs. In this problem, each $C(K_n)$ corresponds with at least one possible distribution of n gifts to $\binom{n}{2}$ guests. Therefore, proving a $C(K_n)$ exists proves that distribution is possible.

Theorem 2.1

Theorem: $\exists\, C(K_n)\, \forall\, K_n$ *such that* $n = 4k$ *or* $n = 4k + 1, k \in \mathbb{Z}^+$

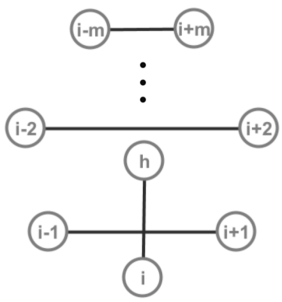
Proof: Consider the two cases for a possible n.

Case 1: $n = 4k + 1, k \in \mathbb{Z}^+$. Isolate one vertex, v_i, from K_n. Consider the remaining 4k vertices and the edges connecting v_{i-1} to v_{i+1}, v_{i-2} to v_{i+2}, and in general v_{i-m} to v_{i+m}, as shown below.



Call this subset of edges ϵ_i. Since each edge in ϵ_i connects 2 of the 4k remaining vertices, there are 2k, an even number of edges in ϵ_i. Additionally, no two edges of $E_{p,i}$ meet at the same vertex. Therefore, $C(\epsilon_i)$ exists with k colors. In terms of n instead of k, there are $n - 1$ vertices and therefore $\frac{n-1}{2}$ edges in ϵ_i. Repeating this process of coloring each ϵ_v with $\frac{n-1}{4}$ colors for all n vertices gives a $C(K_n)$ covering all $\frac{n-1}{2} \times n = \frac{n(n-1)}{2}$ edges with $\frac{n-1}{4} \times n = \frac{n(n-1)}{4}$ colors.

Case 2: $n = 4k, k \in \mathbb{Z}^+$. Rearrange the vertices of K_n so that one vertex, v_h, is in the center and the other 4k-1 vertices are spaced evenly around it. Isolate v_h and one vertex, v_i, from K_n. Consider the remaining 4k-2 vertices and the edges connecting v_{i-1} to v_{i+1}, v_{i-2} to v_{i+2}, and in general v_{i-m} to v_{i+m} along with the edge from v_h to v_i, as shown below.



Again, call this subset of edges ϵ_i. ϵ_i is also a **1-factor** [1], or perfect matching, of K_n, which is a set of edges that includes each vertex of the graph exactly once. There are $\frac{4k-2}{2} = 2k-1$ edges connecting the vertices excluding v_i and v_h, and 1 edge between v_i and v_h, giving 2k, an even number of edges. Additionally, since ϵ_i is a 1-factor, no two edges meet at the same vertex. Therefore, $C(\epsilon_i)$ exists with k colors. In terms of n instead of k, there are n vertices and therefore $\frac{n}{2}$ edges in ϵ_i. Repeating this process of coloring each ϵ_v with $\frac{n}{4}$ colors for all $n-1$ outer vertices gives a $C(K_n)$ covering all $\frac{n}{2} \times (n-1) = \frac{n(n-1)}{2}$ edges with $\frac{n}{4} \times (n-1) = \frac{n(n-1)}{4}$ colors. $\square$

Therefore, since $\exists\, C(K_n) \,\forall\, K_n$ such that $n = 4k$ or $n = 4k+1, k \in \mathbb{Z}^+$, a valid distribution of the n gifts to the $\binom{n}{2}$ guests is always possible, so long as the number of guests is even.

3 Generalizing The Number of Distributions

3.1 General Equation for $\delta(n)$

Definition: $\zeta(K_n)$ is the set of all $C(K_n)$.

Then $|\zeta(K_n)|$ is the number of valid colorings of K_n. In the given problem, for example, $|\zeta(K_n)| = 6$.

However, as seen in the original problem, the number of valid colorings is not equal to the number of valid gift distributions. For each $C(K_n)$, each of the $\frac{n(n-1)}{4}$ couples can be assigned to a color $\frac{n(n-1)}{4}!$ ways, and within each couple the two baskets can be switched between partners, increasing the number of ways by a factor of $2^{\frac{n(n-1)}{4}}$. Therefore, the equation for $\delta(n)$, the number of valid gift distributions for n guests, is:

$$\delta(n) = |\zeta(K_n)| \times \frac{n(n-1)}{4}! \times 2^{\frac{n(n-1)}{4}} \tag{3.1}$$

Now, we must go about generalizing $|\zeta(K_n)|$. This process is complex and thus difficult to enumerate; as n increases, $|\zeta(K_n)|$ increases rapidly. The rest of this section attempts to establish a weak bound on $|\zeta(K_n)|$.

3.2 Accounting for Symmetric Colorings

Recall that in the original problem one coloring was symmetric while the other was not, so the symmetric coloring added 1 $C(K_n)$ while the non-symmetric one added 5, for a total of 6. Another way to think of it is that there were 10 overall possible $C(K_n)$, but 5 were symmetric with each other, so 4 were subtracted. In this case, $4 = S(n)$.

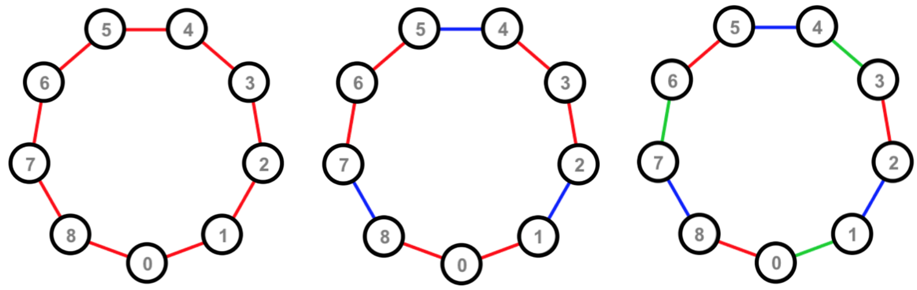
Definition: $S(n)$ is the number of radial symmetrical colorings (colorings such that the rotation of the coloring around the vertices results in the same partition of edges into pairs) over-counted that need to be subtracted from the total colorings to accurately find $|\zeta(K_n)|$

Proposition 3.1 *$\forall(K_n)$ such that $n = 4k$ or $n = 4k+1, k \in \mathbb{Z}^+$, every ϵ_v must have the same $C(\epsilon_v)$, or every $C(\epsilon_v)$ that is different must be evenly spaced around the n outer edges in order for the graph to have complete radial symmetry.*

First of all, E_m, the pair of edges assigned color m must stay in the same in the stay $C(\epsilon_v)$. If they did not, the 2 different $C(\epsilon_v)$ the two edges belonged to would have to be evenly spaced around the graph, which is impossible because in the $n = 4k+1$ case there are an odd number of outer edges and thus rotations, and for the $n = 4k$ case, one vertex can be put in the middle, as shown above, to make there be an odd number of outer edges.

But why does each $C(\epsilon_v)$ have to be the same or evenly spaced?

Consider K_9. For the sake of simplicity, only the outer edges are shown. Let each outer edge be a representative of that edge's $C(\epsilon_i)$. Edges of the same color represent that those two ϵ_v have the same $C(\epsilon_v)$. The three cases of symmetric K_9 are as follows:



In the first case, all $v \in V$ have identical $C(\epsilon_v)$. Thus the graph can be rotated 9 times without creating a new $C(K_9)$, meaning that 8 of these are over counted.

In both the the second and third cases, different $C(\epsilon_v)$ are evenly spaced, so the graph can be rotated 3 different ways without creating a new $C(K_9)$, meaning 2 of these are over counted.

In terms of generalizing, As n gets more factors, the symmetries of K_n get more complex, leading us into number theory. For now, to keep it simple, $S(n)$ will be defined for only n that are prime or of the form 2^k, where ϵ_v cannot be evenly spaced around n outer edges in the $4k+1$ case or $n-1$ edges in the $4k$ case, and thus the only $C(K_n)$ with radial symmetry are ones where every $C(\epsilon_v)$ is identical. Therefore $S(n)$ is the number of $C(\epsilon_v)$ times the number of outside verticies minus 1 symmetries that are over counted. Given an ϵ_v with $|E(\epsilon_v)|$ edges, $|\zeta(\epsilon_v)| = \frac{|E(\epsilon_v)|!}{2^{\frac{|E(\epsilon_v)|}{2}} \times \frac{|E(\epsilon_v)|}{2}!}$ since there are $|E(\epsilon_v)|!$ ways to pick the edges, however edges of the same color can switch places, so this is divided by $2^{\frac{|E(\epsilon_v)|}{2}}$, and additionally entire pairs can swap places, dividing the amount by $\frac{|E(\epsilon_v)|}{2}!$.

Also, recall from Section 2.2 that for $n = 4k+1$, $|E(\epsilon_v)| = \frac{n-1}{2}$, and for $n = 4k$, $|E(\epsilon_v)| = \frac{n}{2}$

Therefore when n is prime or $n = 2^k$:

$$S(n) = \frac{\frac{n-1}{2}!}{2^{\frac{n-1}{4}} \times \frac{n-1}{4}!} \times (n-1)\, when\, n = 4k+1, k \in \mathbb{Z}^+ \tag{3.2}$$

$$S(n) = \frac{\frac{n}{2}!}{2^{\frac{n}{4}} \times \frac{n}{4}!} \times (n-2)\, when\, n = 4k, k \in \mathbb{Z}^+ \tag{3.3}$$

When n is not prime and $n \neq 2^k$, S(n) is larger.

3.3 A Lower Bound on $|\zeta(K_n)|$

An easy way to get a very weak, but definite, lower bound is to consider only $C(K_n)$ where an edge and its color pair must exist within the same ϵ_v. This works well because a ϵ_v partition is the largest partition of K_n such that any pairing of its edges are guaranteed to be a valid coloring since no edges are adjacent. An addition of any edge to ϵ_v must be adjacent to a preexisting edge, adding possible invalid colorings, and therefore over counting valid colorings.

Thus, the number of $C(K_n)$ is simply $|\zeta(\epsilon_v)|$ raised to the number of outside vertices, since each can have any $C(\epsilon_v)$, minus $S(n)$ to account for similarities. Therefore the lower bound can be established:

$$|\zeta(K_n)| \geq (\frac{\frac{n-1}{2}!}{2^{\frac{n-1}{4}} \times \frac{n-1}{4}!})^n - S(n)\, when\, n = 4k+1, k \in \mathbb{Z}^+ \tag{3.4}$$

$$|\zeta(K_n)| \geq (\frac{\frac{n}{2}!}{2^{\frac{n}{4}} \times \frac{n}{4}!})^{n-1} - S(n)\, when\, n = 4k, k \in \mathbb{Z}^+ \tag{3.5}$$

3.4 An Upper Bound on $|\zeta(K_n)|$

Beyond ϵ_v, $|\zeta(K_n)|$ becomes hard to bound. I am working on an upper bound, for now, the highest possible, and thus weakest, yet simplest upper bound is just the number of ways the edges of K_n can be paired up, regardless if a pair's coloring is valid or not. Ideally, I will find a systematic way to subtract these invalid colorings. For now, however:

$$|\zeta(K_n)| < (\frac{\frac{n(n-1)}{2}!}{2^{\frac{n(n-1)}{4}} \times \frac{n(n-1)}{4}!})^{n-1} - S(n) \tag{3.6}$$

4 Conclusion

4.1 Summary

In this paper:

(1) The original problem was solved and explained: The 10 guests can distribute the 10 baskets between them, and this can be done 23040 ways.

(2) It was generalized that for any n gifts and $\binom{n}{2}$ guests, gift distribution is possible between them, so long as the number of guests are even and then able to be in couples.

(3) The number of possible distributions was generalized to be $\delta(n) = |\zeta(K_n)| \times \frac{n(n-1)}{4}! \times 2^{\frac{n(n-1)}{4}}$. However, an equation for $|\zeta(K_n)|$ was not found, instead attempts were made to bound the value and understand the role of symmetries.

4.2 Further Research

With further research, hopefully either $|\zeta(K_n)|$ or a tighter bound on it could be generalized.

Additionally, the original question could have been generalized a number of different ways, for example: What if the guests were not in couples, but triples or quadruples (so much for monogamy)? What if each basket had n gifts instead of 2?

References

[1] Beseri, Tina, and Unal Ufuktepe. "Edge Coloring of A Graph." Diss., Izmir Institute of Technology (2004).

[2] Gross, Jonathan L., and Thomas W. Tucker. Topological graph theory. Courier Corporation, 1987.

[3] Guichard, David. "An Introduction to Combinatorics and Graph Theory." Whitman College (2015).

[4] Weisstein, Eric W. "Complete Graph." From MathWorld–A Wolfram Web Resource.