Building the graph of triangulations of the regular n-gon

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Abstract

This paper will examine the graph of all triangulations of regular polygons. Each vertex on this graph is defined by a unique triangulation of the regular polygon, and the edges represent the ability to get from one triangulation to another via an elementary move or "flip." A flip is when one diagonal is erased from a triangulation and the opposite diagonal is drawn in on the remaining quadrilateral, to get another unique triangulation. Here is the beginning of an algorithm to create the graph of triangulations, where each vertex represents a unique triangulation of the regular polygon and edges represent the ability to get from one triangulation to another via a flip.

1 Introduction

1.1 Definitions: The graph of triangulations

1. An n-***gon*** is a regular polygon with n sides.

2. Given an n-gon, a ***triangulation*** is its division into triangles by drawing non-intersecting diagonals that connect vertices of the polygon. Diagonals will also be referred to as interior edges.

3. A ***diagonal flip***, or elementary move, is the process of erasing one diagonal from a triangulation of the n-gon and replacing it with the opposite diagonal in the quadrilateral created.

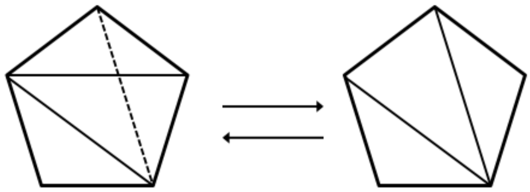

Figure 1: Diagonal flip on a pentagon

4. ***R(n)*** is the graph whose vertices are every distinct triangulation of the n-gon, and whose edges represent the ability to get from one triangulation to another via a diagonal flip. Let V_n denote the vertex set of $R(n)$ and let E_n denote the edge set of $R(n)$.

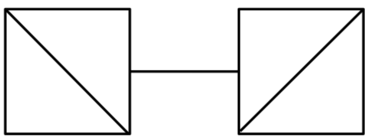

Figure 2: R(4)

1.1.1 Graph theory of $R(n)$

This paper was inspired by the famous and influential paper "Rotation Distance, Triangulations, and Hyperbolic Geometry" by Daniel Sleator, Robert Tarjan, and Bill Thurston, which introduces $R(n)$, but does not talk about any of its properties other than its diameter. $R(n)$ is significant in multiple aspects.

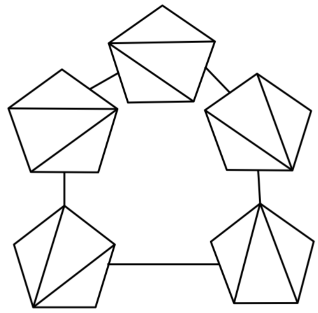

Figure 3: R(5)

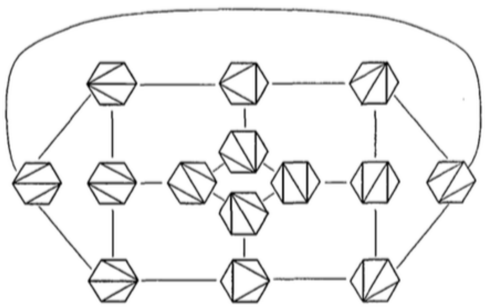

Figure 4: R(6), from [STT]

It has been studied for hundreds of years, most notably by Leonhard Euler [EG]. This graph has many implications in data structures, giving it significance in computer science [STT]. We study it here because it is a beautiful geometric graph with many fascinating symmetries.

Basic Problem:

The basic task I set forth to accomplish was to gain an understanding of some quantitative and qualitative graph-theoretic properties of $R(n)$. One example of a property of this graph to be analyzed is its diameter, or the longest distance between two of its vertices/triangulations. Sleator, Tarjan and Thurston proved that the diameter of $R(n)$ is exactly $2n-10$ for any n-gon.

In order to discover properties of $R(n)$, I first wanted to gather experimental data about $R(n)$ for various n. Once I began sketching R(7), my hand was tired, and I became interested in another, related question: how do I create this graph? More specifically, how can I tell a computer to produce $R(n)$ so that one can perform experimental data to discover graph theoretic properties?

2 An Algorithm to Build $R(n)$

Problem: Give an algorithm for a computer to create $R(n)$

Input: $n \geq 3$

Output: $R(n)$

The algorithm would assign labels to the vertices of $R(n)$ $1, 2, ..., n$, such that each number represents a specific vertex, and output the pairs of vertices $\{(x_1, y_1), ...(x_r, y_r)\}$, representing the edges of the graph.

2.1 Finding the Vertices of $R(n)$

Euler conjectured the theorem that determines the number of vertices of $R(n)$ in a letter to Christian Goldbach in 1751. 87 years later in 1838, Gabriel Lamé elegantly proved this theorem in a letter to Joseph Liouville.
The following proof is due to Lamé [La].

Recall that V_n denotes the vertex set of $R(n)$; in other words, it is the set of triangulations of the n-gon. Let N_n denote $|V_n|$; in other words the total number of triangulations of an n-gon.

Theorem 2.1 $N_{n-1}+N_3N_{n-2}+\cdots+N_{n-k+2}N_k+\cdots+N_{n-2}N_3+N_{n-1} = N_n$, *for* $n \geq 3$ *and* $n \geq k \geq 3$.

Proof:

Let numbers label the vertices of an n-gon, letting $12345...n$ represent a convex polygon with n sides. An arbitrary side 12 of $12345...n$ is the base of a triangle in each N_n triangulations of the n-gon. The triangle has a third vertex in each triangulation at $3, 4, 5, 6...n$. For triangle 123, there will correspond N_{n-1} different triangulations, as the vertex "2" will be isolated from the rest of the n-gon. For triangle 124 there will be N_3N_{n-1} different triangulations; for triangle

125 there will be $N_4 N_{n-2}$ (meaning the pairing of each distinct triangulation of N_4 with each distinct triangulation of N_{n-2}); and finally for triangle $12(n)$, which will again correspond to N_{n-1} triangulations. Each of these triangulation groups is distinct, meaning their sum gives N_n. □

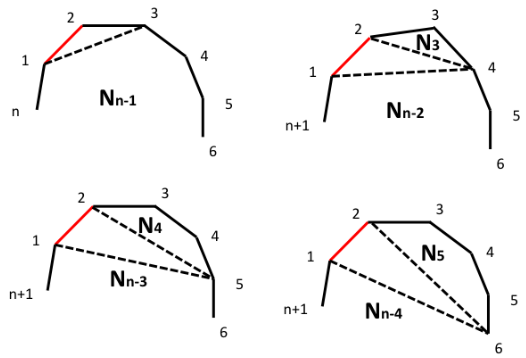

Figure 5: A schematic showing the triangulation types used in the proof of Lamé's theorem 2.1.

We can use the method described in this proof to algorithmically determine all of the vertices of $R(n)$. An interior edge in the triangulation exactly corresponds to an unordered pair (a, b) with a $\neq b$.

Proposition 2.2 *Any triangulation of a regular n-gon has $n-3$ interior edges for $n \geq 3$.*

We proceed by induction.

Base case: The 3-gon has $3-3=0$ interior edges in every triangulation.

Assume by induction a triangulation of a regular n-gon has $n-3$ interior edges for all $n \leq k$.

Want to show: The number of interior edges of $(k+1)$-gon$=(k+1)-3$.

In an arbitrary triangulation of the $(k+1)$-gon, isolate an interior edge, thus splitting the triangulation of the $(k+1)$-gon into two distinct pieces. Call one piece of the triangulation an a-gon and the other a b-gon. So, $(a-1)+(b-1)=$

$a + b - 2 = k + 1$.

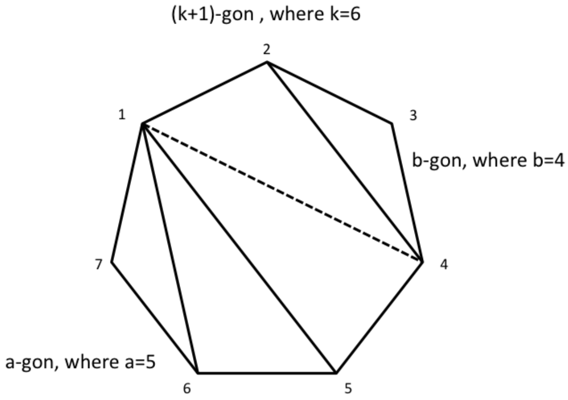

Figure 6: $k + 1$-gon, split into an a-gon and b-gon, where $k = 6$, $a = 5$, and $b = 4$

By our inductive hypothesis, the b-gon has $b-3$ interior edges, and the a-gon has $a - 3$ interior edges.

Therefore, adding one for the isolated edge, the number of interior edges of the $(k + 1)$-gon=$(a - 3) + (b - 3) + 1 = a + b - 5$.

Substituting $k+1$ for $a+b-2$, the number of interior edges of the $(k+1)$-gon $(k + 1) - 3$. $\square$

$R(n)$ is connected [STT]. This provides another proof of Proposition 2.2, as follows. The triangulation N_n given by $\{(1, 3), (1, 4), (1, 5), \ldots, (1, n - 1)\}$, where each ordered pair represents an interior edge between vertices of the n-gon, has $n - 3$ interior edges, as vertex 1 cannot connect to itself, vertex 2 or vertex n.

Because this graph is connected, and flips maintain the number of interior edges of the polygon, every triangulation of the n-gon will have $n - 3$ interior edges. $\square$

Therefore, any triangulation determines a set of $n-3$ distinct unordered pairs $\{(a_1, b_1), (a_2, b_2), \ldots, (a_{n-3}, b_{n-3})\}$, where each interior edge of the triangulation gives (a_i,b_i). Note that not every possible set of $n-3$ pairs occurs. There are some obvious restrictions:

If (a, b) occurs then $|a-b| > 1$. If $(a, a+2)$ occurs, then $(a+1, b)$ does not occur for any b.

A triangulation of an n-gon can be encoded by a set of $n-3$ pairs of numbers $\{(a_1, b_1), (a_2, b_2), \ldots, (a_{n-3}, b_{n-3})\}$ where the edges of the triangulation are exactly given by the line segment from a_i to b_i.

For each triangulation N_n, each triangulation of N_k can be matched with each triangulation of N_{n-k+2}, for $n \geq 3$ and $n \geq k \geq 3$. This matching done as shown in Lamé's proof of theorem 2.1. Hence, we have the vertices of $R(n)$.

2.2 Finding the edges of $R(n)$

Here is a useful formula for finding the number of edges of $R(n)$. Using this number, we can determine when the algorithm to build $R(n)$ is complete, as it tells us when all of the edges are built. Recall that N_n denotes the number of vertices of $R(n)$.

Proposition 2.3 *The number of edges of* $R(n) = \frac{N_n * (n-3)}{2}$

Proof:

We showed previously that the number of interior edges of an n-gon is $n-3$. An edge in $R(n)$ is defined by the ability to get from one triangulation to another via a diagonal flip, meaning that a single flip defines an edge. Therefore, there are $n-3$ flips that can be performed on each triangulation, meaning the degree of each vertex of $R(n) = n-3$ (which means $R(n)$ is an $(n-3)$-regular graph). Therefore, the number of edges of $R(n)$ is given by:

$$\frac{N_n * (n-3)}{2}$$

dividing by two because each edge was counted twice. □

Now, we will look at the three steps needed to determine the edges of $R(n)$.

Step one: Expressing the vertex set, called V_n, of $R(n)$ as a disjoint union of vertices, $\Gamma_3, \Gamma_4, \ldots, \Gamma_i, \ldots, \Gamma_n$. The proof in §2.1 of the number of vertices in V_n, and the algorithm we used to generate V_n, gives a way to express V_n as a disjoint union. We will use the labels from §2.1. Each Γ_i is characterized by the ith vertex of the n-gon used to create the triangle $12i$ in the n-gon's triangulation.

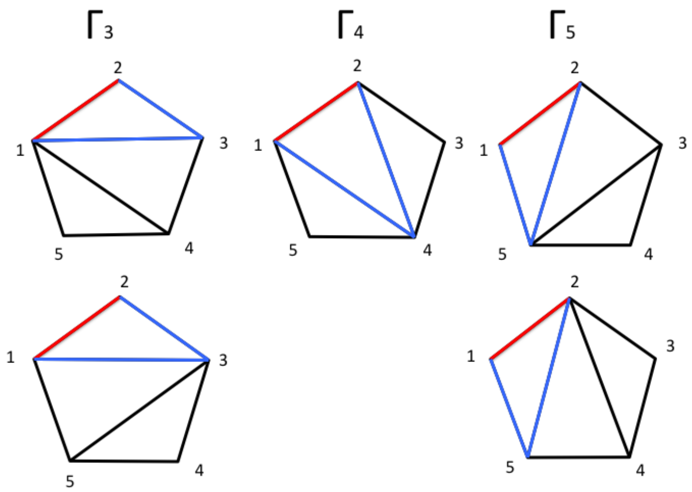

Figure 7: This diagram gives an example of the vertices of the 5-gon that lie in Γ_3, Γ_4 , and Γ_5.

Notice the symmetry in the number of vertices in the Γ_i subsets for the 5-gon, as shown in Figure 2.2. The line of symmetry lies in Γ_4, or the middle Γ_i, in this example. This is a property that will hold true for all n, because, for example, Γ_n is simply a rotation of Γ_3. This rotational symmetry exists because, in both Γ_3 and Γ_n, the n-gon is split into a 3-gon and an $(n-1)$-gon.

Step two: Finding the edges within each Γ_i. For each vertex Γ_i, the $n-gon$ is split into an i-gon, a triangle $12i$, and an $(n-i+2)$-gon. Let's call a triangulation T_x.

The vertices in Γ_i are defined by $\{(T_1, T_2) : T_1 \in R(i) \text{ and } T_2 \in R(n+i+2)\}$. (T_1, T_2) and (T_2, T_4) are connected by an edge if and only if:

$$T_1 = T_3 \textbf{ and } T_3 \text{ and } T_4 \text{ are connected by an edge in } R(n-i+2),$$

or

$$T_2 = T_4 \textbf{ and } T_1 \text{ and } T_3 \text{ are connected by an edge in } R(n-i+2).$$

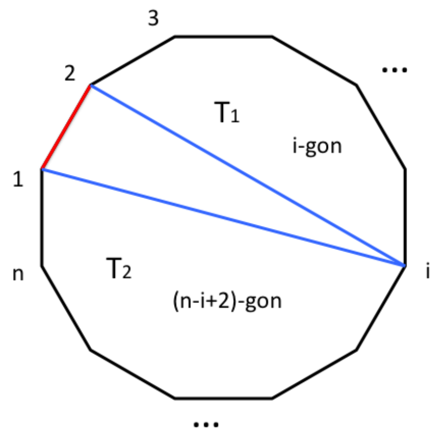

Figure 8: A schematic representing Γ_i of the n-gon, and the triangulations of the i-gon and the $(n-i+2)$-gon that it is comprised of.

In other words, there is an edge between two vertices in Γ_i if and only if the triangulations of the i-gons for each vertex differ by one diagonal flip AND the triangulation of the $(n-i+2)$-gon for each vertex is the same, OR the triangulations of the $(n-i+2)$-gons for each vertex differ by one diagonal flip AND the triangulations of the i-gon for each vertex is the same.

This is true because for each triangulation of the n-gon, the i-gon and $n-i+2$-gon are completely separated by the triangle $12i$, so two pieces must be identical while the other two pieces differ by one flip in order for the two triangulations to differ by exactly one flip.

Another way of thinking about it is every Γ_i is defined by the graph $R(i)$, with each vertex representing some triangulation of $R(i)$, the triangle $12i$, paired with the graph $R(n-i+2)$. There is a one to one correspondence between vertices in each subgraph for which the paired triangulations of $R(i)$ are connected by an edge, for which the triangulation of the $(n-i+2)$-gon are identical. This means that there are $(n-i+2)$ edges between subgraph.

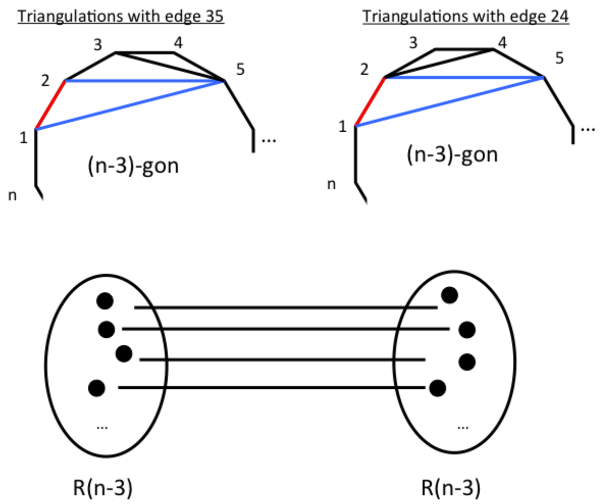

Figure 9: This diagram is a representation of Γ_4 for the n-gon, where each dot represents a vertex of $R(n)$. There are N_{n-3} edges between vertices in each oval.

Step three: Find the edges between every Γ_i. I have some conjectures about this behavior, namely that the same number of edges connect Γ_1 and Γ_2 as connect $\Gamma_n - 1$ and Γ_n. Experimental data on $R(6)$ demonstrated that this symmetry in the edges does not occur between rotationally symmetric vertices. I have more conjectures that I am still working on.

3 Conclusion

3.1 The answered questions about $R(n)$

1. Is it connected? Yes [STT]

   What is its diameter? $2n-10$ [STT]

2. Number of vertices? (Recall N_n is the number of vertices of $R(n)$)

$N_{n-1} + N_3N_{n-2} + \cdots + N_{n-k+2}N_k + \cdots + N_{n-2}N_3 + N_{n-1} = N_n$, for $n \geq 3$ and $n \geq k \geq 3$

3. Number of edges?

$$\frac{N_n * (n-3)}{2}$$

4. Algorithm to build $R(n)$?

   We showed how to construct the vertices of $R(n)$, and the first two steps of how to connect these vertices to create edges. Once we fully understand how to find the edges of R(n), our algorithm can be applied to many n-gons on a computer to gather tons of data on $R(n)$, in order to better understand it and its graph theoretic properties.

3.2 Open Questions

1. Apply problem to triangulations of the Taurus.

2. We saw above that any triangulation of N_n determines a set of n-3 distinct unordered pairs $(a_1$,$b_1)(a_2$,$b_2)$...$(a_{n-3}$,$b_{n-3})$, where each edge of the triangulation gives of of the (a_i, b_i). We gave a few restrictions on which such sets can occur. Are there any equations to describe which sets occur?

References

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