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# Sanitary Surgeons

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**Abstract**

In Martin Gardner's book "AHA", there is a problem in which there is a hospital and a certain number of surgeons who have to operate on a patient. The surgeons and the patient may be contaminated. Contamination can spread from examination without gloves and from a person to a side of the glove or from one glove to another touching glove. We want to find the minimum number of gloves needed so that all surgeons operate and no one gets contaminated from someone else. The gloves used are reversible. On first glance, the answer might seem obvious. But on closer examination, it becomes a very intriguing puzzle.

## 1. Introduction

In a hospital, there are a certain number of surgeons. They all have to operate on a patient, who may be contaminated. The surgeons themselves may be contaminated. We want to ensure that all the surgeons operate on the patient without somebody being contaminated from another person. Contamination can spread from examination without gloves and from a person to a side of the glove or from one glove to another touching glove. Since there is a short supply of gloves, the surgeons want to use the fewest number of gloves possible. Gloves are reversible and can be worn on top of each other. So with all this information, what is the least number of gloves needed for $n$ surgeons to operate on one patient? Also, what about if we have multiple patients? How many gloves are needed for each of $n$ surgeons to operate on each of $k$ patients?

## 2. Multiple surgeons with 1 patient

Let us start by making things simple. Let us assume that every surgeon only has one hand, instead of two. The number of gloves needed will equal the number of pairs needed if surgeons have 2 hands. We can notate gloves as following: $G$ stands for a glove. $G_k$ means that it is glove $k$. $G_k^1$ means glove $k$, side $1$ and $G_k^2$ means glove $k$, side $2$. Our first problem is this:

**Problem 1.** Let $k(n)$ be the minimum number of gloves needed for n surgeons to operate on 1 patient so that nobody get contaminated from another person. Find $k(n)$.

**Result 1.** $k(n) \leq n$.

*Proof.* Each of the surgeons will operate with a different glove and the patient will touch the opposite sides of the gloves. Each of the surgeons operates and as no one touches a contaminated side or person, everybody is safe.

After this result, I was not getting too far, so then, I reversed the problem. If there are n gloves, what is the maximum number of surgeons that can operate? We can define the following:

**Definition 1.** Let $j(n)$ to be the maximum number of surgeons that can all operate using n gloves nobody get contaminated from another person.

**Result 2.** $j(n) \geq 2n - 2$.

*Proof.* Let us assign one side of one glove to always touch the patient. Let it be $G_n^2$. The first surgeon can wear $G_1^1$. On top of this, he can wear $G_n^1$. The process repeats with the $d$th surgeon wearing $G_d^1$, with $G_n^1$ on top of it. We keep on going till surgeon $n-1$. We don't have any more gloves. Now the remaining $n-1$ surgeons can reverse gloves and wear sides $G_1^2$, $G_2^2$, ..., $G_{n-1}^2$, respectively. So for example, surgeon $n-1+h$ would wear $G_h^2$. And each surgeon will wear $G_n^1$ on top. We would do this till we reached surgeon $2n-2$, at which we would have no more clean sides. The patient only touches $G_n^1$ and this side isn't touched by any other glove. For all other gloves, one side $(G_1^1, ..., G_{n-1}^1)$ is touched only by one surgeon each and the other side touches uncontaminated $G_n^2$ and then only one surgeon each.

**Result 3.** $j(n) \geq 2n-1$.

*Proof.* We can use the method in the proof of Result 2. When we look carefully, we realize that all surgeons wear glove $G_n$ on top and nobody touches the side $G_n^1$. We can have another surgeon wear this. But this side can get contaminated when after $n-1$ surgeons have operated, other surgeons wear reversed gloves. So after $n-1$ surgeons have operated, a new surgeon can wear just glove $G_n$ touching the side $G_n^1$. This will not contaminate the remaining $n-1$ surgeons as they will wear reversed gloves on sides $G_1^2$, $G_2^2$, ..., $G_{n-1}^2$ and these sides touched the side $G_n^1$ of the outer glove in first $n-1$ operations but will not touch it any more. So there is no risk of contamination.

**Result 4.** $j(n) = 2n-1$.

*Proof.* We need to prove that $j(n) < 2n$, that is, $n$ gloves are not enough for $2n$ surgeons. There are $2n$ sides of $n$ gloves. To be completely safe, every surgeon and patient have to have their own side of a glove. With $2n$ surgeons and $1$ patient, $2n+1$ sides are needed which is more than the total $2n$ sides of $n$ gloves.

**Result 5.** $k(2n-1) = n$ and $k(2n) = n+1$.

*Proof.* From Result 4, $n$ gloves are sufficient for $2n-1$ surgeons but $n-1$ gloves are sufficient for only $2n-3$ surgeons so for $2n-1$ surgeons, a minimum of $n$ gloves are needed and $k(2n-1) = n$. From Result 4, $n+1$ gloves are sufficient for $2n+1$ surgeons so they are also sufficient for $2n$ surgeons. But $n$ gloves are not sufficient for $2n$ surgeons because $n$ gloves are sufficient for only

$2n-1$ surgeons. So $k(2n) = n+1$.

## 3. Multiple Patients

But what about multiple patients? Let $p(n, m)$ be the minimum number of pairs of gloves needed for each of the $n$ surgeons to operate on each of the $m$ patients. Let us start with $2n-1$ surgeons and $1$ patient. Result 3 shows that $n$ gloves are needed. If we add another patient, we will need another glove because all $2n$ sides of the $n$ gloves are contaminated. We make the following assumption.

**Assumption 1.** *One surgeon can complete surgeries on all patients before another surgeon operates.*

*How do we solve this problem? We can start by assuming that surgeons can use the same side of the glove for all patients. That means that $n-1$ gloves, $G_1$ to $G_{n-1}$ will be reused for all patients.*

**Result 6.** $p(2n-1, m) \leq n+m-1$.

*Proof.* The glove $G_n$ that doctors wear on top of one glove to operate on patient $1$ can not be used to operate on patient $2$. A new glove must be worn on top for Patient $2$. Let us add one. The new glove will be touched by the clean side of $n-1$ gloves when first $n-1$ surgeons operate, and then will be touched by the contaminated sides. This is because each surgeon can do all of his/her surgeries first, and then give the gloves to the other surgeon who needs them. If this wasn't the case, we could not have the $n$th surgeon work at all as all the sides would be contaminated. So for $2n-1$ surgeons and $m$ patients, we would need the original $n$ gloves, and then we would need $1$ for each of the other $m-1$ patients after the first. This leads us to having a total of $n+m-1$ pairs of gloves for $2n-1$ surgeons and $m$ patients.

If the number of surgeons is even, say $2n$, the number of gloves needed is at most the number of gloves needed when the number of surgeons is $2n-1$. This gives the following result.

**Result 7.** $p(n, m) \leq \lfloor \frac{n+1}{2} \rfloor + m - 1$.

The number of gloves needed equals about half the number of surgeons plus the number of patients. But surgeons and patients are similar because all of them can get contaminated and come in contact with each other. So in theory surgeons and patients can be switched. But in our formula, the number of gloves increases by $1$ when two surgeons are added, but increases by $2$ when $2$ patients are added. So the number of gloves increases more with number of patients. This seems inefficient when there are a large number of patients. So we can switch the role of surgeons and patients in the above method. Then, for $n$ surgeons and $2m-1$ patients, it is sufficient to use $n+m-1$ gloves. We can use the minimum of the number of gloves needed in these two methods.

**Result 8.** $p(n,m) \leq \min(\lfloor \frac{n+1}{2} \rfloor + m - 1, \lfloor \frac{m+1}{2} \rfloor + n - 1)$.

While we have made some progress on the second problem with multiple surgeons and multiple patients by finding an upper bound on $p(n,m)$, we have not been able to prove that the above result gives the minimum number of gloves needed. We also know that the lower bound on $p(n,m)$ is $(m+n)/2$. The upper bound and the lower bound are not close but we think $p(n,m)$ cannot be very close to the lower bound but we cannot prove it.

## 4. Conclusion

We have solved the problem with one patient and multiple surgeons (or one surgeon and multiple patients). We have also found upper bounds on the number of gloves needed with multiple surgeons and multiple patients. The number of gloves needed is never more than the number of people and can be almost half of the number of people if there are many more surgeons than patients or many more patients than surgeons. We have not been able to solve the problem exactly. The optimal solution may require wearing multiple gloves on top of each other. The problem is complicated because the sequence in which the operations are carried out can also have an effect on passing contamination. Martin Gardner's book stated that the problem has not been solved. Also, I did not find a solution anywhere else. This problem may extend into graph theory, as if we express everything as points in an organized way, and then have the way gloves are used for one surgeon as a line and connect all the points. But it is difficult to express timing of operations in a graph. For further research on this topic, I will go deeper on this link with graph theory. I also might add twists such as not all the gloves are reversible.

## Bibliography

Gardner, Martin. Aha!: Aha! Insight. New York: Scientific American, 1978. Print.