Dividing Circles into Portions- Part 2

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Abstract: I have a circle and a certain number of straight lines. The objective is to divide the circle into as many portions as possible with the lines. I've found an interesting correlation between the number of lines used and the greatest number of portions last year, but this year I am continuing my project- what would happen if I put a hole in the circle? Or two?

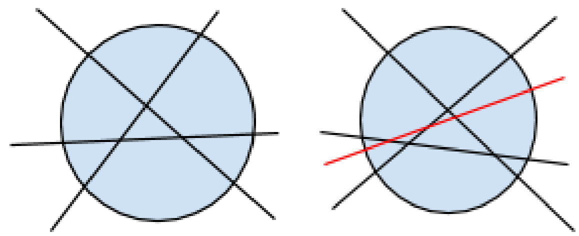
Introduction: I first found this problem in a MATHCOUNTS competition, and have expanded it since. Dividing circles into portions is mundane, but once the cuts are arranged in a certain way, you could divide it into more portions than your average pizza or pie.

Body: I started my project with a regular circle, and dividing it into different portions.

Lines	**Portions**
0	1
1	2
2	4
3	7
4	11
5	16
6	22

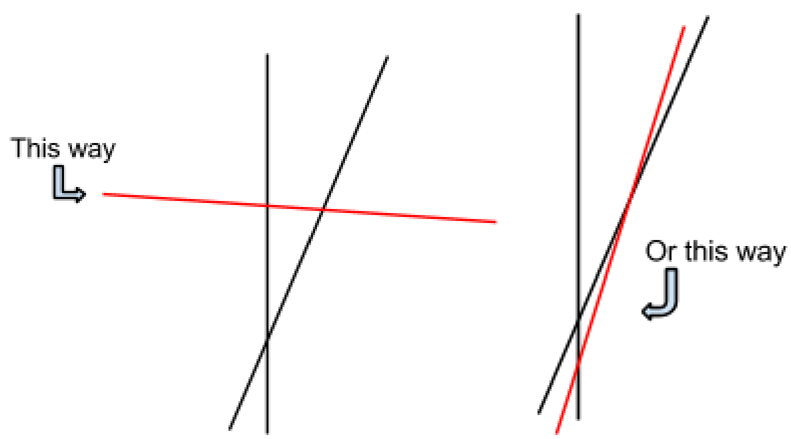
I found many fascinating things. In my previous study, I found:

- The number of portions a new cut, or line, adds to the number of preexisting portions is equivalent to the total number of lines in the circle, as seen here:



The circle on the left, call it 0-3 (because it has 0 holes and 3 lines) has 7 portions. The next one, 0-4, has 4 lines (1 extra) and 11 portions. The new line adds 4 more portions to the portions already there, and there are 4 lines. This is an interesting pattern.

- To yield the greatest number of portions, each line must cross all the other lines. This is pretty self-explanatory, as more lines crossed makes more portions. A line cannot cross another line twice because lines are straight.
- One line could always intersect all the other lines:

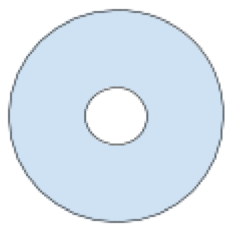


And there would never be a point where you cannot fit more lines, because lines have no width, only length, and you could get infinitely smaller with the portions.

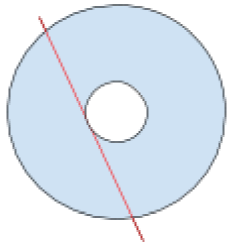
The formula for finding the number of portions for any given number of lines n in a circle with 0 holes is: $\frac{1}{2}\left(n^2+n+2\right)$

This is what I found last year. This year, I looked for a master rule for circles with any number of holes h and any number of lines n.

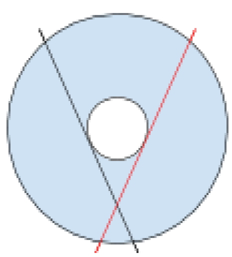
I started with one hole. The circle is still 1 portion.



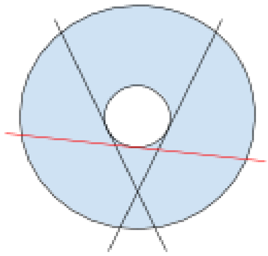
1 line: 2 portions



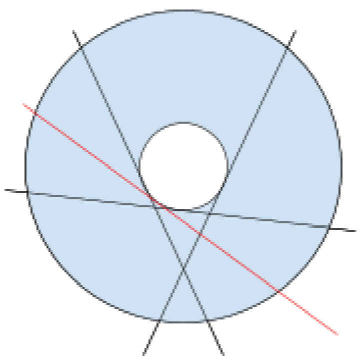
2 lines: 5 portions, instead of the 4 for the circle with 0 holes. This is the first time there is a difference in hole and no hole.



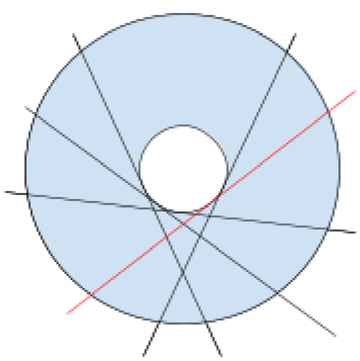
3 lines: 9 portions. Notice how the new line (in red) still intersects all the other lines, even when there is a hole- so this is the greatest number of portions.



4 lines: 14 portions. By now I have realized to produce the greatest number of portions in a circle with a hole, the line needs to touch the hole. This gives it the effect in both circles 1-3 and 1-4, producing two extra portions using the edge of the hole.



5 lines: 20 portions.



And finally, 6 lines: 27 portions.

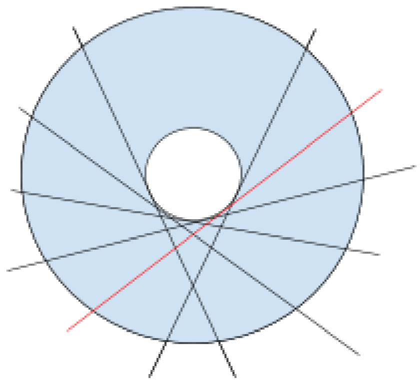


Table:

Lines	**Portions**
0	1
1	2
2	5
3	9
4	14
5	20
6	27

It can be observed here that the positive difference between each consecutive portion number increases by 1 starting with 3, the positive difference of the portion numbers of circles 1-1 and 1-2.

2 5 9 14 20 27

∨ ∨ ∨ ∨ ∨

3 4 5 6 7

∨ ∨ ∨ ∨

1 1 1 1

The tree diagram above leads to our rule for circles with 1 hole: $\frac{1}{2}\left(n^2+3n\right)$

Checkstep 1: Where $n=1$: $\frac{1}{2}\left(1^2+3\times 1\right)=\frac{1}{2}\times 4=2$

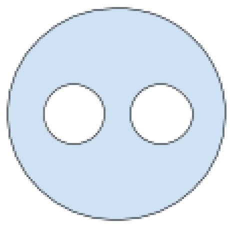
Checkstep 2: Where $n=2$: $\frac{1}{2}\left(2^2+3\times 2\right)=\frac{1}{2}\times 10=5$

Checkstep 3: Where $n=3$: $\frac{1}{2}\left(3^2+3\times 3\right)=\frac{1}{2}\times 18=9$

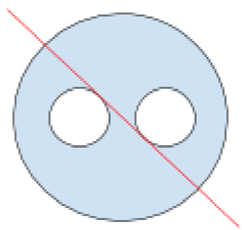
Checkstep 4: Where $n=4$: $\frac{1}{2}\left(4^2+3\times 4\right)=\frac{1}{2}\times 28=14$

Now, let us start on two-hole circles.

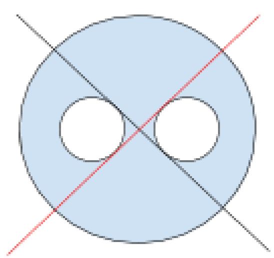
0 lines: 1 portion.



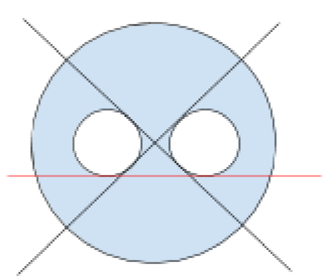
1 line: 2 portions.



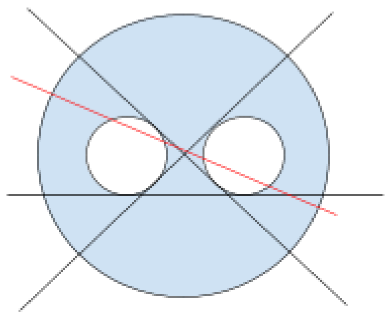
2 lines: 6 portions. I had reasoned that my discovery for one-hole circles, that to yield the greatest number of portions a new line had to intersect all other lines and touch the hole at the same time, worked also for two-hole circles.



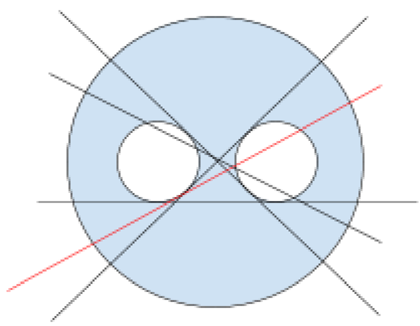
3 lines: 11 portions.



4 lines: 17 portions. This one was difficult, because I could not figure out how to intersect 3 lines and touch 2 holes. I finally found that by cutting across the holes the greatest number of portions could be achieved.



5 lines: 24 portions.



6 lines: 32 portions.

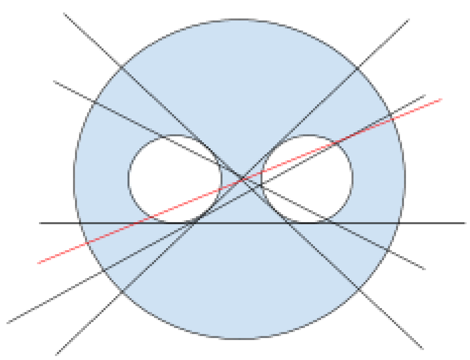


Table:

Lines	**Portions**
0	1
1	2
2	6
3	11
4	17
5	24
6	32

As with the circles with 1 hole, the positive difference between each consecutive portion number increases by 1, this time starting with 4, the positive difference of the portion numbers of circles 2-1 and 2-2 as shown in the tree diagram below.

2 6 11 17 24 32
V V V V V
4 5 6 7 8
V V V V
1 1 1 1

The rule for 2-hole circles, then, is: $\frac{1}{2}\left(n^{2}+5n-2\right)$

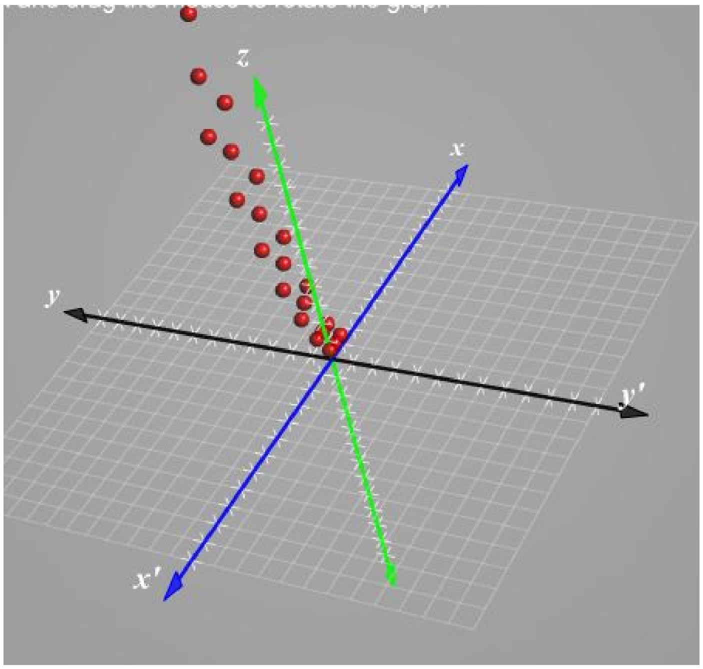
Checkstep 1: Where $n=1$: $\frac{1}{2}\left(1^{2}+5\times 1-2\right)=\frac{1}{2}\times 4=2$

Checkstep 2: Where $n=2$: $\frac{1}{2}\left(2^{2}+5\times 2-2\right)=\frac{1}{2}\times 12=6$

Checkstep 3: Where $n=3$: $\frac{1}{2}\left(3^{2}+5\times 3-2\right)=\frac{1}{2}\times 22=11$

Checkstep 4: Where $n=4$: $\frac{1}{2}\left(4^{2}+5\times 4-2\right)=\frac{1}{2}\times 34=17$

My goal now is to find a rule for circles with any number of holes h and any number of lines n. I was stuck on this for a while. At first I thought if I made it into a coordinate graph with h being x, n being y, and the greatest number of portions, p being z:



But I couldn't find an equation for that 3-D coordinate graph either. So I started listing out some coordinates, trying to find a correlation:

0, 5, 16

1, 5, 20

2, 5, 24

0, 2, 4

1, 2, 5

2, 2, 6

0, 4, 11

1, 4, 14

2, 4, 17

As you can see, I made the n-value constant with every 3 h-values. And the pattern I saw was incredible. The positive difference between the p in each consecutive h was $n-1$. For example, the last three listed: (0, 4, 11), (1, 4, 14), and (2, 4, 17). The positive difference between 11, 14, and 17 was 3, or $4-1$. Which means that I could simply add $h(n-1)$ to the first circle, the one with 0 holes, and get my number of portions. My master rule, then, is this:

$$\tfrac{1}{2}\left(n^{2}+n+2\right)+h(n-1)$$

Where $\tfrac{1}{2}\left(n^{2}+n+2\right)$, remember, is the equation I found to the greatest number of portions for the circle with 0 holes. Here are my checksteps:

0 holes, 5 lines: $\tfrac{1}{2}\left(5^{2}+5+2\right)+0(5-1)=16$

1 hole, 4 lines: $\tfrac{1}{2}\left(4^{2}+4+2\right)+1(4-1)=14$

2 holes, 6 lines: $\tfrac{1}{2}\left(6^{2}+6+2\right)+2(6-1)=32$

2 holes, 3 lines: $\tfrac{1}{2}\left(3^{2}+3+2\right)+2(3-1)=11$

Conclusion: The major thing I learned from continuing my project from last year is that there are more sides to a problem than we think. There are infinitely many possibilities to all problems, and all problems are ever-expanding. I found the master rule for greatest number of portions for any number of lines with any number of holes in a circle, $\tfrac{1}{2}\left(n^{2}+n+2\right)+h(n-1)$, which is a pretty impressive discovery, if I do say so myself.

One major problem I encountered in this project was the finding of the master rule, as I mentioned in the body. Another problem was figuring out how to produce the greatest number of lines for circles with holes- an entirely new side to the problem I did in 7th grade. Especially difficult was the 2-hole circle; by the time I got to the 4th line I had no choice but to cut across the holes in order to touch both of them.

I think I'm done with this problem now. Continuing it was fun, but further exploring would be repetitive. In 7th grade I asked the question of whether it would change if the shape was not a circle at all, but a square or triangle- which brings me back to my original point, that all problems have infinitely many sides. Kind of like a circle. And it's impossible for one to explore all of them. I guess we can only try.