The Missing Box

A Counting Problem

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Abstract

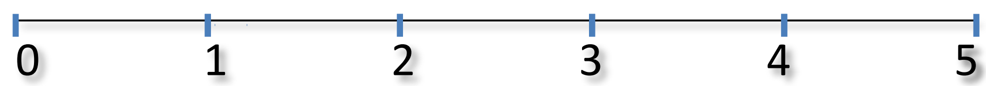
In this paper, I count the number of boxes that can be made inside of a larger box made from unit cubes with one cube missing to answer the question, "Where could I remove a unit cube but maximize the number of smaller boxes I can make?" To do this, I use constructive and complimentary counting and then derive a generalized formula. I experiment with different locations of missing boxes and show that each location follows the formula's prediction for the number of boxes one is able to create. In the future, this counting method could be applied to different shapes or a fourth dimension.

Introduction

How can counting boxes be so tricky? We learn to count in kindergarten, right when we learn numbers. We learn addition and multiplication, subtraction and division. We learn powers of ten, fractions, and factorials. And then we take a class on counting and combinatorics. One would have good reason to argue that all he learned in math class is how to count. But, counting is more interesting than one might think. Counting can be frustratingly difficult, especially when something is too numerous to count, or too hard to count one-by-one. In the summer of 2017, Dr. Harold Reiter from UNC Charlotte challenged my class to count the number of boxes that can be made within a larger box made of unit cubes. This paper will take his problem a step further: I will attempt to count the number of smaller boxes that can be made when one unit cube is missing from the center, edge, or corner of a 3x3x3 box. I hypothesize that the number of smaller boxes that can be made with increase as the missing box moves toward the corner. I will also venture to derive a generalized formula for counting small boxes, and the journey starts with a number line.

Counting boxes in 1 dimension

To attempt this problem, we must start with the mechanics of box counting. And to explain this, let's begin with a section of a number line.



Now let's define a 1D box as a segment in a section of the number line where the endpoints are integers. For example, the section in the figure is from 0 to 5. Possible segments in that case are from 2 to 3, or from 0 to 5, but not from 1 to 6 (because that section of the line is not completely included in the one from 0 to 5) and not from 2 to 2 (because that is not a 1D box, it is a 0D box). So, a 1D box can be defined by two different integers between 0 and 5 where the box spans the length from one integer to the other. So, there are 6 choose 2, or 15 boxes, because the 6 represents the number of integers between 0 and 5 inclusive.

Counting boxes in 2 and 3 dimensions

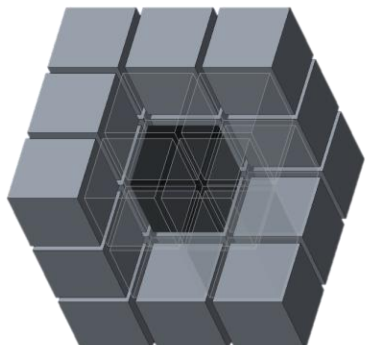
Now let's count the smaller boxes in a 5x5 2D box. We will say these boxes are defined by the line segments on the bottom and right of the box. And, we can say those are defined by 4 numbers: two from 0 to 5 in the *x*-axis, and two from 0 to 5 in the *y*-axis, so there are $\binom{6}{2}^2 = 225$ boxes.

Now, it is simple to count the smaller boxes in a 5x5x5 3D box – a smaller box is defined by 3 lines, those along the x-, y-, and z-axis. So, there are $\binom{6}{2}^3 = 3375$ boxes. We have solved Dr. Reiter's problem.

Counting boxes if a unit cube is missing

The box we want to analyze in this way has a hole in it: represented by the black small cube.

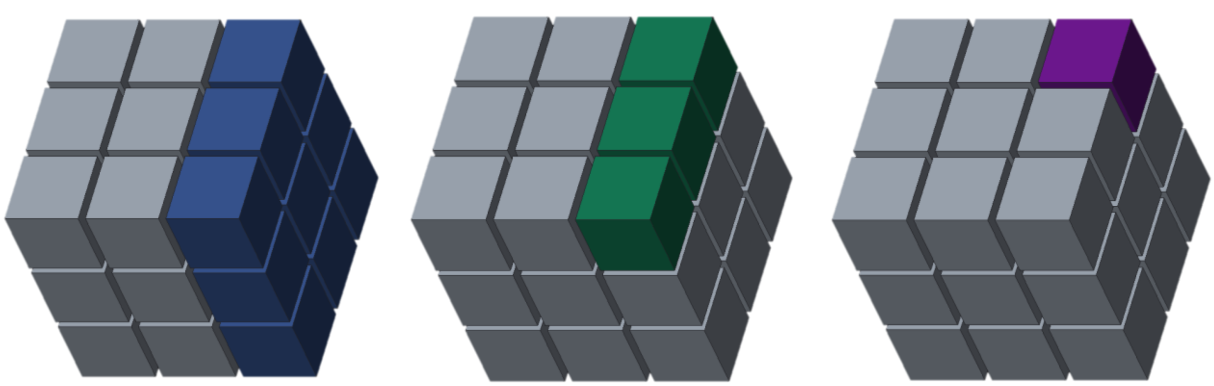
To find the number of smaller boxes in this cube, where the smaller box does not contain the



hole, I chose to use the power of complimentary and constructive counting. So, let's define.

A "section" will be produced from a straight cut to the side of the hole in the cube, an "edge" will be a continuous volume of the cube where two or more sections intersect, and a "corner" will be a continuous volume of the cube where exactly three sections intersect. By these definitions, all corners are parts of edges, which are parts of sections.

The following diagrams show examples of sections, edges, and corners on a cube with empty core. The blue represents one section, the green an edge, and the purple a corner.



So, after all this definition, we are finally able to find a formula for the number of smaller boxes in our cube with an empty core.

Building the cube

First, let us start with the front section. How many smaller boxes can be fit inside it? The answer is, simply $\binom{4}{2}^2 = 36$, just by using our formula for 2D boxes. Next, let us add the right section. Because there is an overlap between the front and right section (defined as an edge), we will use complimentary counting to determine the number of smaller boxes added by completing the right section. This turns out to be $\binom{4}{2}^2 - \binom{4}{2} = 30$, using our methods for 2D (right face) and 1D (edge) boxes. Similarly, the back face should use the same formula as the right face, so it is again $\binom{4}{2}^2 - \binom{4}{2} = 30$. The left face overlaps two of our already counted faces, producing two edges that we have to subtract, so the formula is $\binom{4}{2}^2 - 2\binom{4}{2}$ = 24. So far, we have the boxes in 4 sections minus the boxes in 4 edges.

Now, how many boxes are in the top face? We have four extra edges this time, so one would think the formula would be $\binom{4}{2}^2 - 4\binom{4}{2}$. However, in subtracting the edges in the top face, we subtracted the corners twice, so we have to add them back. One can only make one box in a 1x1x1 corner , and there are 4 corners on the top face. This makes the formula for the top face $\binom{4}{2}^2 - 4\binom{4}{2} + 4(1) = 16$. The same works for the bottom face. Therefore, in all, we have 6 sections minus 12 edges plus 8 corners making $6\binom{4}{2}^2 - 12\binom{4}{2} + 8(1) = 216 - 72 + 8 = 152$ small boxes in total (Appendix A).

This method can be generalized. So, our grand formula for the total number of boxes when a center cube is missing (t) is

$$s - e + c = t$$

where s is the sum of all of the boxes that can be made in a section, e is the sum of all of the boxes that can be made from the edges, and c is the sum of all of the boxes made from the corners.

Missing corner cube

This formula works in the first cube, but will it work if the corner unit cube is missing? To find out, we must take a careful look at sections, corners, and edges. Again, the 3x3x3 cube is deconstructed with cuts along each side of the missing cube.

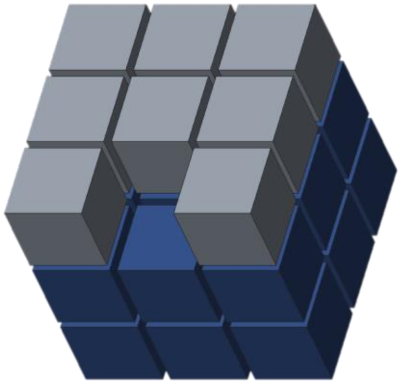
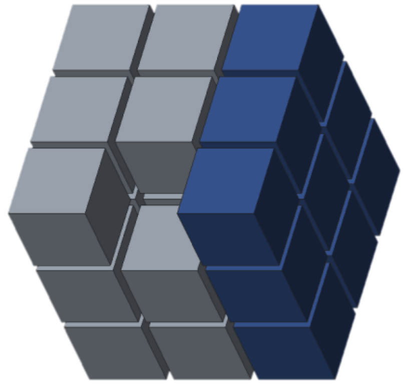
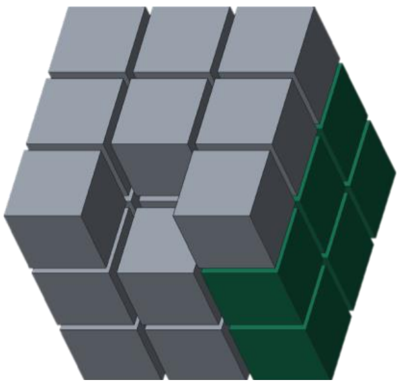
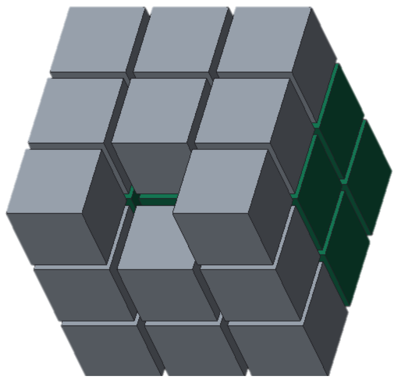
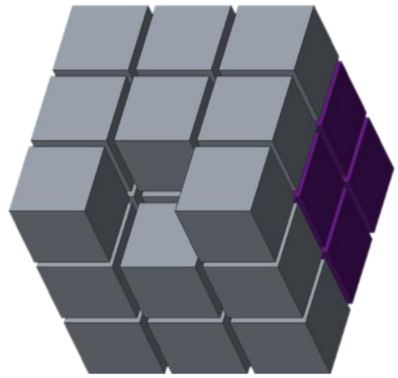
We have three sections in this new cube, each touching the three imaginary inside faces of the missing corner cube. These sections extend to the three faces opposite the nonexistent corner, which means their dimensions are 2x3x3: a right section (colored in blue above), a front section, and a bottom section. The overlap between the three sections defines the one and only corner (shown in purple). Where any two sections overlap, there is an edge produced (the intersection of the front and right sections is colored in green above).

Because there are three sections here, and each pair of them can produce an edge, there are $\binom{3}{2}$ = 3 edges that will be present on the cube of size 3x2x2. These edges correspond to the 3 buried edges of the missing cube. So, we now have found 3 sections, 3 edges, and 1 corner, and we are ready to count constructively and compare the results to our formula.

We will use the same complimentary and constructive counting method that we used before on the empty core. Before, we started with a section; so we will begin with our 3x3x2 section now, and there are $\binom{4}{2}^2\binom{3}{2} = 108$ smaller boxes. We added another section and subtracted its corresponding edge, and we will duplicate that maneuver presently, making $\binom{4}{2}^2\binom{3}{2} - \binom{3}{2}^2\binom{4}{2} = 108 - 54 = 54$ boxes. However, because this time we will only have one section left, we will subtract the two edges formed and then, we will have to add the corner that results from the intersection of those edges, making $\binom{4}{2}^2\binom{3}{2} - 2\binom{3}{2}^2\binom{4}{2} + \binom{3}{2}^3 = 108 - 108 + 27 = 27$ boxes added in completing the final face. This gives us $108 + 54 + 27$ boxes total, or 189. By following our previously derived formula, we have 3 sections – 3 edges + 1 corner, or $3\binom{4}{2}^2\binom{3}{2} - 3\binom{3}{2}^2\binom{4}{2} + \binom{3}{2}^3 = 324 - 162 + 27 = 189$ boxes in total, the same answer (Appendix B).

Missing edge cube

Let us see if the formula also works on a box with a missing edge cube. Here, we have four sections, one for every inside face of the missing cube. To investigate the dimensions of the sections, we can just take the very first cube we analyzed (with the missing center) and push the missing box first to the side and then to the top. In this process, two sections vanish, and the opposite sections elongate to double their original size. Because the other two sections kept their dimensions, two of the sections are 3x3x1 and two are 3x3x2 (see two example sections in blue below).

The edges also vary in proportion, but it is not hard to determine their extent. There are two types of intersections: 3x3x1 section intersecting a 3x3x2 section and a 3x3x2 intersecting a 3x3x2. The first one, where a smaller section intersects a bigger section, creates four edges of magnitude 1x2x3 (represented above by the bottom left cube). The two larger sections intersect to form one 2x2x3 edge (represented by the bottom center cube) making five edges in total corresponding to the five buried edges of the missing cube.

From the four sections, we can produce two corners. Each of these corners is the intersection of two larger sections with face with one smaller section (shown in purple). This makes the two corners 1x2x2, so now we are ready to build this third cube.

To build this cube, we start with the bottom section (in this case, a 3x3x2), having $\binom{4}{2}^2\binom{3}{2} = 108$ boxes. And then, we add the side sections, one by one. For the first side section, we subtract a 1x2x3 edge, making the number of new boxes created $\binom{4}{2}^2 - \binom{4}{2}\binom{3}{2} = 36 - 18 = 18$. The opposite side section adds the same number of new boxes. For the rear large section, there are three other sections it intersects: the bottom large section, and two small sections. These produce one 2x2x3 edge and two 1x2x3 edges, but these edges intersect, so we have to add back the two corners. So there are $\binom{4}{2}^2\binom{3}{2} - \binom{3}{2}^2\binom{4}{2} - 2\binom{3}{2}\binom{4}{2} + 2\binom{3}{2}^2 = 108 - 54 - 36 + 18 = 36$ boxes added in that last face. This makes 108 + 18 + 18 + 36 = 180 boxes. Using the formula derived from the missing center cube, we have: $\left[2\binom{4}{2}^2\binom{3}{2} + 2\binom{4}{2}^2\right] - \left[4\binom{4}{2}\binom{3}{2} + \binom{3}{2}^2\binom{4}{2}\right] + 2\binom{3}{2}^2 = 288 - 126 + 18 = 180$. Again, our formula works! (Appendix C)

Difficulties encountered while solving this problem

Finding what type of counting to use was tough. The first time I solved Dr. Reiter's problem, I didn't use combinations. Instead, I found how many boxes could be made with their tops as part of the very top layer of the bigger box. And then how many with their tops as part of the next-to-top part of the big box, and so on. On summation, these numbers gave the correct answer; but this method would be inefficient for the more complex problem which is the subject of this paper. That is when I turned to simplifying the problem to a 1D box, and found combinations efficient.

Another difficulty I encountered while solving this problem was visualizing three-dimensional cubes, especially with holes in them. Finding cross-sections was a very helpful way to see these complex things, but sometimes you just need a 3D modeling program, such as Pro/ENGINEER which was used to create the figures in this paper.

Conclusion

Our formula for the number of smaller boxes that can be made from a box with a hole in it is $s - e + c = t$, where s is the number of boxes that can be made from the sections, e is the number of boxes that can be made from the edges, c is the number of boxes that can be made from the corners, and t is our answer. To derive this formula, we built three cubes with a missing unit cube in three different places. We used constructive and complimentary counting to do this, and in support of my hypothesis, as the hole got closer to the corner, t increased. I believe this can be used for all cubes and proven by induction.

Now that we have this formula, we can find other things, such as a probability of having a box that includes the center or even how to pack boxes when arranging around one special unit box. We could also experiment on a tetrahedron, octahedron, or even a 4D box. We can also find the number of boxes on surface areas.

Acknowledgements

I'd like to thank Dr. Harold Reiter for inspiring me to investigate this problem, Jānis Lozovskis for his advice, and Math Circles of Chicago.

Appendix A

missing center cube

division	count	dimensions	combination	total boxes
sections	6	3x3x1	$\binom{4}{2}^2 = 36$	6 x 36 = 216
edges	12	3x1x1	$\binom{4}{2} = 6$	12 x 6 = 72
corners	8	1x1x1	1	8 x 1 = 8
sections - edges + corners =				**152**

Appendix B

missing corner cube

division	count	dimensions	combination	total boxes
sections	3	3x3x2	$\binom{4}{2}^2 \binom{3}{2} = 108$	3 x 108 = 324
edges	3	3x2x2	$\binom{3}{2}^2 \binom{4}{2} = 54$	3 x 54 = 162
corners	1	2x2x2	$\binom{3}{2}^3 = 27$	1 x 27 = 27
sections - edges + corners =				**189**

Appendix C

missing edge cube

division	count	dimensions	combination	total boxes
sections	2	3x3x2	$\binom{4}{2}^2\binom{3}{2} = 108$	216
	2	3x3x1	$\binom{4}{2}^2 = 36$	72
edges	4	3x2x1	$\binom{3}{2}\binom{4}{2} = 18$	72
	1	3x2x2	$\binom{3}{2}^2\binom{4}{2} = 54$	54
corners	2	2x2x1	$\binom{3}{2}^2 = 9$	18
sections - edges + corners =				**180**