Which Game Features Affect Your Ability to Master *Mastermind*?

Mathematics

Design Investigation

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(Signature of Sponsoring Teacher)

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ABSTRACT

Mastermind is a combinatorics game where a code-maker competes against a code-breaker. Our project examines how three variables of Mastermind - the number of colors, whether colors can repeat, and whether the order of the colors matters - affect the number of possible arrangements that the code-maker can create. We investigate how these three variables affect the difficulty of Mastermind for the code-breaker.

We identify twelve variations of Mastermind and compute the number of possible arrangements available to the code-maker for each variation. We also describe the four formulas that we used to find these computations.

Our results show that the factor that affects the game of Mastermind the most is whether order matters. The factor that has the next largest impact on the game is whether colors can repeat. The number of colors matters the least. However, all three variables positively affect the number of possible arrangements available to the code-maker (and this affects the difficulty of the game for the code-breaker).

HYPOTHESIS AND PURPOSE

Mastermind is a code-guessing game that involves combinatorics and skill. In our project, we vary three factors: the number of colors (“Color”), whether colors may repeat (“Repeats”), and whether the order of the colors matters (“Order”). We hypothesized that all three variables would affect the number of possible arrangements available to the code-maker. From our basic knowledge of combinatorics, we anticipated that Order would affect the game more than Color and Repeats. We then predicted that Color and Repeats would affect Mastermind similarly.

Mastermind is a game that we have both played numerous times, and that we both thoroughly enjoy. The idea of combinatorics interested us because we wanted to understand more about how the game of Mastermind works and which factors - Colors, Repeats, and Order - make Mastermind hard to master.

INTRODUCTION: HISTORY AND RULES OF MASTERMIND

Mastermind’s History

Mastermind is a board game that has been in existence for almost 50 years. It was first invented by Mordecai Meirowitz, an Israeli postmaster and telecommunicator, in 1970. The game was an adaption of the much older paper game known as “Bulls and Cows,” which had similar characteristics (Waeber, "Mastermind Game"). It also may have been inspired by moo, a program developed by J. M. Grochow at MIT in the late 1960s (Weisstein, “Wolfram MathWorld”). The idea, which was rejected by many large publishing companies, finally was picked up by Invicta Plastics, which had rights to the game after buying it from Meirowitz.

Following its introduction in 1970 in the United States, Israel, and the United Kingdom, the game has now sold over 55 million copies of the board game across the globe!

How to Play Mastermind

Mastermind is a code-breaking game for two players, a code-maker and a code-breaker. The code-maker chooses four colors from a pool of six different colors to build a code. The code-maker may use the same color more than once. The code-breaker has ten tries to guess the 4-color code. Following each guess, the code-maker provides feedback to the code-breaker using black and white pegs. When the code-maker responds with a black peg, that signifies that the code-breaker has guessed a color and it is in the corresponding position in the actual code. When the code-breaker receives a white peg, the code-breaker guessed a correct color, but it is not in the correct position in the actual code. The code-maker does not specify which peg corresponds with which color the code-breaker guessed. This is what makes the game hard and fun. If the code-breaker "breaks the code" in 10 or less moves, they are victorious. Otherwise, it is a victory for the code-maker. Another way of playing is to solve your opponent's code in fewer turns than it takes your opponent to solve your code (Mastermind Board Game, 2019). (In this version, each player has a chance to be the code-maker and the code-breaker.) *In the current Mastermind game, there are 6 possible colors, repeated colors* are *allowed, and order* does *matter.*

Research Questions

We studied how many ways the code-maker can arrange colors under different variations of the game. These different variations concern (a) the number of colors that may be used (Color), (b) whether or not colors can be repeated (Repeats), and (c) whether or not the order in which the colors placed matters (Order). Moreover, we studied how different variations of the game affect the number of possible arrangements available to the code-breaker. Our overall research question is: How do various features of Mastermind affect the number of arrangements/codes in the game that the code-maker can create? We can break this down into three specific sub-questions…

> *Sub-Question 1:* How does the number of colors affect the number of possible arrangements?
> *Sub-Question 2:* How does whether colors can repeat or not affect the number of possible arrangements?
> *Sub-Question 3:* How does order affect the number of arrangements?

METHOD

There are four different equations when calculating the number of possible arrangements for the twelve variations of Mastermind that we are studying. The equations are for: permutations without repetition, permutations with repetition, combinations without repetition, and combinations with repetition. Note that the number of colors affects one of the variables in the equation, not the equation itself. Below, Table 1 shows the permutation and combination formulas.

Table 1: Permutation and Combination Formulas Without Repetition	
Permutation Formula: $_nP_r$	**Combination Formula: $_nC_r$**
$\frac{n!}{(n-r)!}$	$\frac{n!}{r!(n-r)!}$
We use this formula to calculate the number of arrangements when the order ***does*** matter. This formula assumes that objects *cannot* be repeated.	We use this formula to calculate the number of arrangements when the order ***does not*** matter. This formula assumes that objects *cannot* be repeated.

The permutation formula is used when the order *does* matter and the combination formula is used when the order *doesn't* matter. (Lane, *Permutations and Combinations*)Since the code [Red, Green, Blue, Black] is not the same code as [Black, Blue, Green, Red], the game Mastermind is played where the order of colors *does* matter. We used the combination formula when we studied how changing the order so that it doesn't matter affects the game. We observed the two formulas and eventually figured out how they are derived. We divide $_nP_r$ (the permutation formula) by r! to get the total amount of combinations of n objects taken r at a time, $_nC_r$. $_nC_r$ = n!/ ((n-r)! r!) (Taylor, 2018). After constant observation, we discovered that the reason why $r!$ is divided through the permutation formula is because $r!$ is the number of ways that each arrangement can be arranged. For example if [Blue, Red, Black, Yellow] is the code, within itself, it can be arranged 4! times since the permutation and combination formula assume that colors can't repeat (Charles, Illingworth, McNemar, Mills, Ramirez, Reeves, 2010). If we think of each code as its own set, you can make r! number of arrangements for each combination because each of the sets have four 'slots'. The first slot can be any of the four colors. The second slot can only be one of the three remaining colors since one color was already used. The third slot uses one of the remaining two. Finally, the fourth slot only can be the color that it left. Since combinations remove the worth of $r!$ we divide ${}_nP_r$ by $r!$ to get the total number of combinations of *n* objects taken *r* at a time, ${}_nC_r$. (The Editors of Encyclopaedia Britannica, 2018).

For example if we want to figure out the number of possible arrangements for 6 colors, *no* repeats, and when the order *does* matter, we would use the permutation formula where n=6 (the number of colors) and r=4 (the number of slots). When we simplify 6!/(6-4)! we get 360 possible arrangements. To get the number of combinations, we would divide 360 by 4! (r!) which would reduce the number of possible arrangements to 15.

For the variations of Mastermind where colors *can* repeat, we used another two formulas shown below in Table 2.

Table 2: Permutation and Combination Formulas With Repetition	
Permutation Formula: ${}_nP_r$	**Combination Formula: ${}_nC_r$**
n^r	$\frac{(n+r-1)!}{r!(n-1)!}$
We use this formula to calculate the number of arrangements when the order ***does*** matter. This formula assumes that objects *can* be repeated.	We use this formula to calculate the number of arrangements when the order ***does not*** matter. This formula assumes that objects *can* be repeated.

The formulas above are the formulas we used for the permutation and combination variations where colors can be repeated. The permutation with repetition formula is n^r. The two variables of interest are *n* (the number of colors) and *r* (the number of slots the colors can be in). There are 6 colors in the original game and there are 4 slots where the colors can be. Therefore, n=6 and r=4. Since more than one color could be used, *each* slot can be one of six colors.

Therefore, the number of possible arrangements is 6x6x6x6 which is 6^4. In the terms of variables, 6^4 can be shown as n^r.

We were able to understand and derive the general formulas for permutations with and without repetition, and for combinations without repetition. The general formula for combinations with repetition goes beyond the principles that we understand or that are even reviewed in math textbooks that cover counting principles. However, we were able to find a few websites online that provided the general formula. We used this formula to compute the number of possible arrangements for the versions of Mastermind where colors can repeat, and the order doesn't matter.

For these four variations, *n* is always represented as the number of colors. When we study the effect of changing the number of colors, we are replacing *n* with the new number of colors that we are studying and keeping r as 4 since we are not changing the number of slots. For example, when we study the variations with 7 colors, *n* would equal 7.

RESULTS

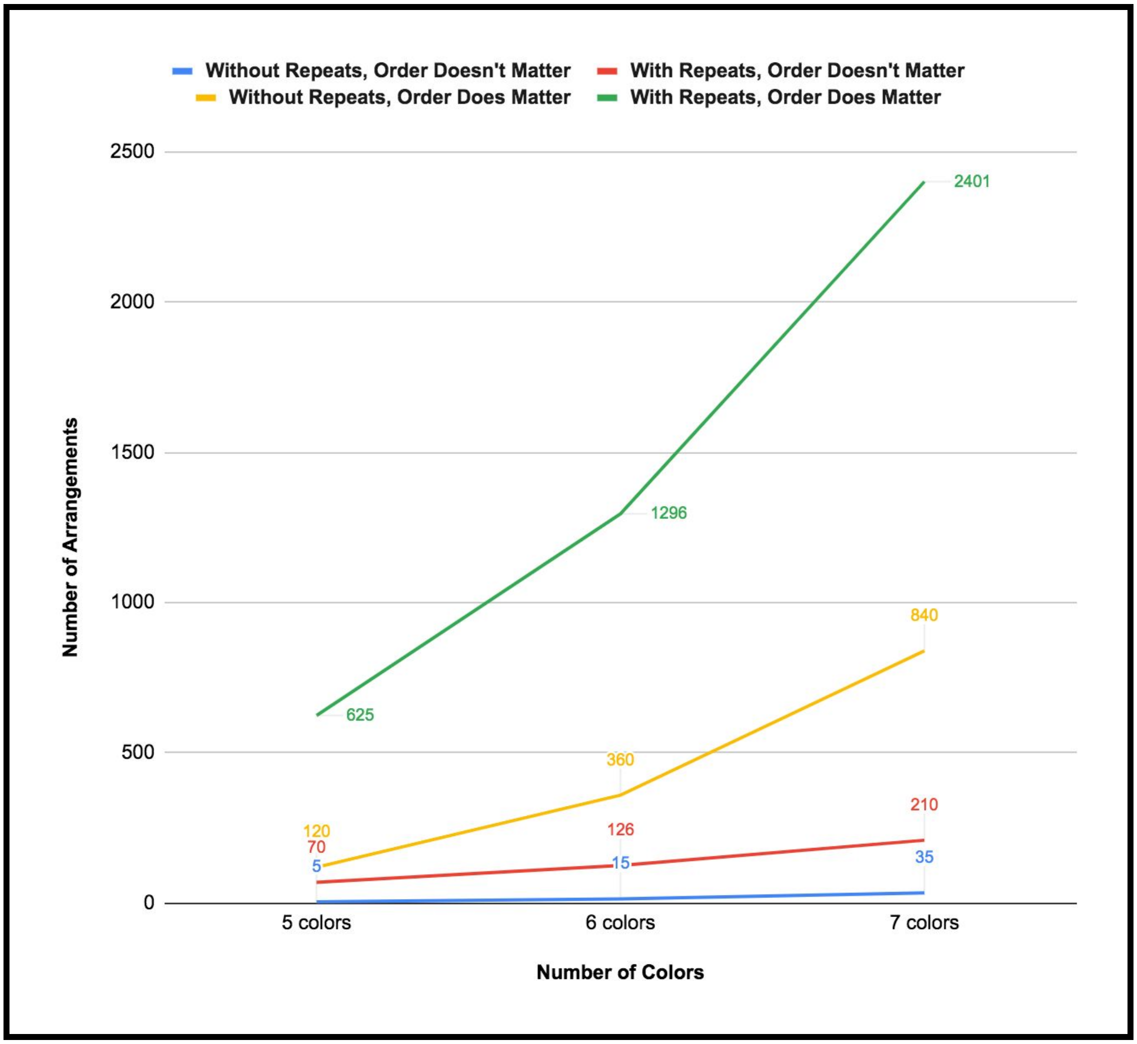
Table 3 (below) shows the percent change in the number of possible arrangements from the current game of Mastermind. By doing this, we saw which game type affected the game the most and built a foundation towards deciphering which variable has the most profound effect on the game. The original game has 6 colors, repetition of colors is allowed, and order matters. From the twelve variations of Mastermind we studied, the original game had the second highest possible arrangements. We can see that the gamemode when there are 5 colors, no repetition, and order doesn't matter is the most effective variation at reducing the number of arrangements, to a mere number of 5. Relatively, the gamemode where there are 7 colors, repetition of colors is allowed, and order matters is the only variation that increases the number of possible arrangements compared to the original game. It is apparent that the introduction of combinations to the game affects it most as we see a trend of very high percentage decreases when the order isn't relevant.

Table 3: The Number of Arrangements and Percent Change from the Original Game for Each Variation

	# of Arrangements	% Change in # of Arrangements from Current Game
5 Colors, Without Repeats, Order Doesn't Matter	5	99.6% decrease
6 Colors, Without Repeats, Order Doesn't Matter	15	98.8% decrease
7 Colors, Without Repeats, Order Doesn't Matter	35	97.3% decrease
5 Colors, Without Repeats, Order Does Matter	120	90.7% decrease
6 Colors, Without Repeats, Order Does Matter	360	72.2% decrease
7 Colors, Without Repeats, Order Does Matter	840	35.2% decrease
5 Colors, With Repeats, Order Doesn't Matter	70	94.6% decrease
6 Colors, With Repeats, Order Doesn't Matter	126	90.3% decrease
7 Colors, With Repeats, Order Doesn't Matter	210	83.8% decrease
5 Colors, With Repeats, Order Does Matter	625	51.8% decrease
6 Colors, With Repeats, Order Does Matter➡Current Game	1296	------------------
7 Colors, With Repeats, Order Does Matter	2401	85.3% increase

Figure 1 summarizes our project by representing all of the retrieved data in a single visualized form. Notice that all trends are positive. This means that as the number of colors increases, repetition is allowed, and the order of the colors matter (permutation variations), the number of possible arrangements increase. This makes the game harder for the code-breaker since there are a higher number of possible arrangements.

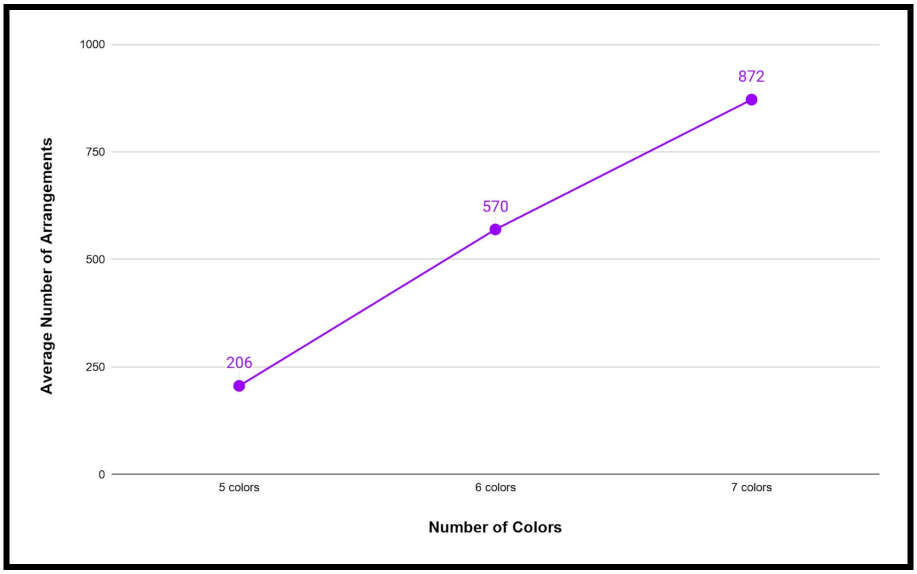
Figure 1: The Effect of All the Variables on the Number of Arrangements



Our results were found by collapsing over variables to isolate how one factor affected the original game of Mastermind. To do this, we averaged the number of arrangements between different specified variables (e.g. collapsing over permutations and combinations would mean averaging the two when there are five colors, six colors, and seven colors). All of the percent changes in our results are measured as percent increases from a lower number to a higher number to make sure they can all be easily compared to the game.

Figure 2 (below) represents the data where we collapsed over combinations and permutations, as well as the variable of repetition or no repetition. This was to isolate the effect of the number of colors on the game of Mastermind in a graph so that we could study its effect on the game and use the visual to make observations on the data we found. From this graph, we can see that there is a clear positive trend as colors increase. The increase from 5 colors (206) to 6 colors (570) was an increase of 177%, and the increase from 6 colors (570) to 7 colors (872) was 53%, which averages to 115% increase between the colors on average. We can also visibly identify that the percentage changes are not equal in size as we can see the slope from 5 to 6 colors is greater than the slope from 6 to 7 colors. However, in terms of pure numbers, the change from 6 to 7 colors (302) was greater than the change from 5 to 6 colors (364).

Figure 2: The Effect of the Number of Colors on the Number of Possible Arrangements



In Figure 3 (below), we collapsed across colors and order to isolate the variable of repetition in the game. We can see that there is again a positive slope, and there is a 260% increase in the number of arrangements possible from 222 to 800. This graph shows that the game played with repetition will have more arrangements than a game played without repetition. Consequently, a variation of Mastermind with repetition is theoretically more difficult than a variation without repetition.

Figure 3: The Effect of Whether Colors Can Repeat on the Number of Possible Arrangements

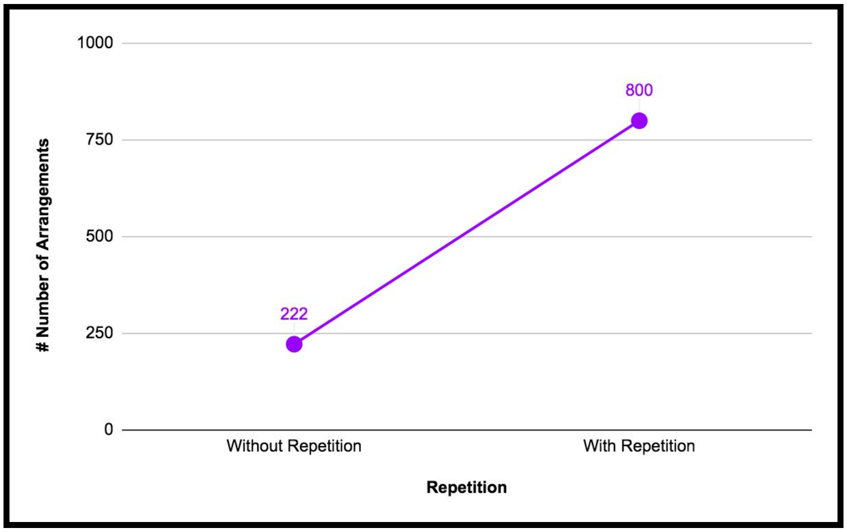
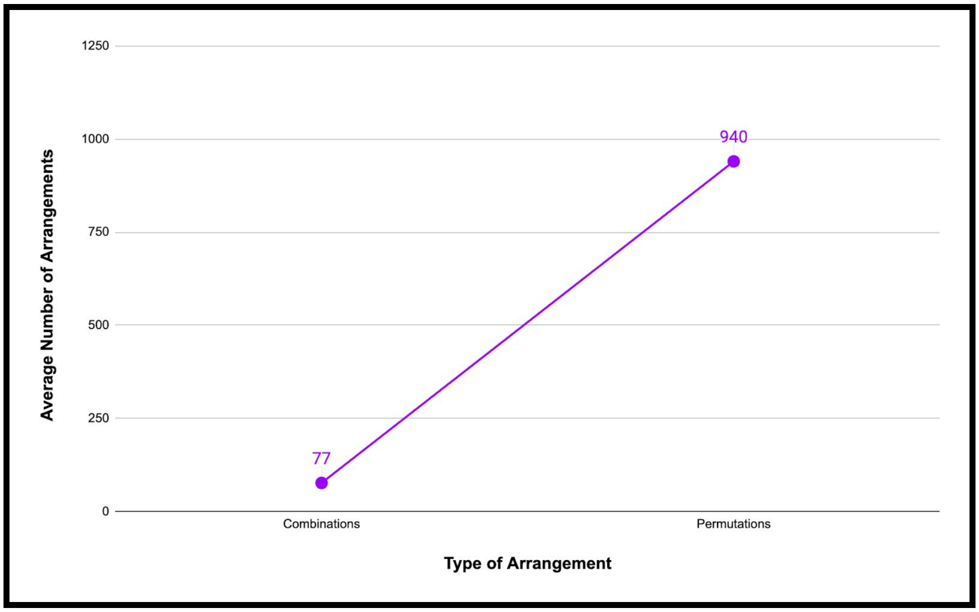


Figure 4 (below) is a visual representation of the data we received when we collapsed over two variables, those being colors and repeats. Whether the order of the colors matters affects the number of arrangements greatly because Combinations are the other side of the spectrum. In a combination game, the order of the colors in a guess does not affect the chances of getting the guess correct. This means a great reduction in the number of arrangements from permutations. In the graph, we noticed a positive trend and a massive increase in the number of arrangements when we compared the two. From 77 arrangements in combinations to 940 in permutations, there was a 1121% increase in the number of arrangements. This shows that combinations and permutations affect the game of Mastermind the most when changed. The original game of Mastermind is based around permutations (where the order of the colors do matter). By changing this variable to combinations, we reduce the number of possible arrangements by 863 arrangements.

Figure 4: The Effect of Order on the Number of Possible Arrangements



CONCLUSION

After collapsing across the three variables to isolate the effect of each variable on the number of possible arrangements, we concluded that all variables - Color, Repeats, and Order - affect the number of possible arrangements in Mastermind. We concluded Order affected Mastermind the most with a whopping 1121% increase in the number of possible arrangements from combinations (when order doesn't matter) to permutations (when order does matter). The second most important variable was Repeats since there was a 260% increase from no repeats to repeats. Colors mattered the least since the average percent increase from color to color was only 115%. Changes in each variable changed the number of possible arrangements and the difficulty of Mastermind. The actual Mastermind has 1296 possible arrangements. The game could be made more difficult by increasing the number of colors. It could be made easier by decreasing the number of colors, by not allowing repeated colors, or by changing the game so that order doesn't matter.

However, this isn't the end. There is so much more we can delve into regarding the game of Mastermind such as different variables' affect on the number of possible arrangements or strategies for winning.

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